1.LIMIT OF A FUNCTION

Lets discuss what a function is

A function is basically a rule which associates an element with another element.

There are different rules that govern different phenomena or happenings in our day to day life.

For example,

i. Water flows from a higher altitude to a lower altitude
ii. Heat flows from higher temperature to a lower temperature.
iii. External force results in change state of a body(Newton's 1st Rule of motion) etc.

All these rules associates an event or element to another event or element, say , x with y.

Mathematically we write,

$$y = f(x)$$

i.e. given the value of x we can determine the value of y by applying the rule 'f'

for example,

$$y = x + 1$$

i.e we calculate the value of y by adding 1 to value of x. This is the rule or function we are discussing.

Since we say a function associates two elements, x and y we can think of two sets A and B such that x is taken from set A and y is taken from set B. Symbolically we write

x ε A (x belongs to A)

y ε B (x belongs to B)

y = f(x) can also be written as

(x,y) ε f

Since (x,y) represents a pair of elements we can think of these in relations

f $\subseteq$ A X B or

f can thought of as a sub set of the product of sets A and B we have earlier referred to.

And, therefore, the elements of f are pair of elements like (x,y).

In the discussion of a function we must consider all the elements of set A and see that no x is associated with two different values of y in the set B

What is domain of function

Since function associates elements x of A to elements y of B and function must take care of all the elements of set A we call the set A as domain of the function. We must take note of the fact that if the function can not be defined for some elements of set A , the domain of the function will be a subset of A.

Example 1

Let A = $\{1,2,3,4,-1,0,-4\}$

B = $\{0,1,2,3,4,-1,-2,-3\}$

The function is given by

y = f(x) = x + 1

for x=1, y= 2

x=2, y=3

x=3,y=4

x=4,y=5

x=-1,y=0

x=0,y=1

x=-4,y=-3

clearly y=5 and y= -3 do not belong to set B. therefore we say the domain of this function is

the set $\{0,1,2,3,-1,\}$ which is a sub set of set A.

What is range of a function

Range of the function is the set of all y's whose values are calculated by taking all the values of x in the domain of the function. Since the domain of the function is either is equal to A or sub set of set A, range of the function is either equal to set b or sub set of set B.

In the earlier example,

Range of function is the set $\{1,2,3,4,0\}$ which is a sub set of set B

SOME FUNDAMENTAL FUNCTIONS

Constant Function

Y = f(x)=K, for all x

The rule here is: the value of y is always k, irrespective of the value of x

This is a very simple rule in the sense that evaluation of the value of y is not required as it is already given as k

Domain of 'f' is set of all real numbers

Range of 'f' is the singleton set containing 'k' alone.

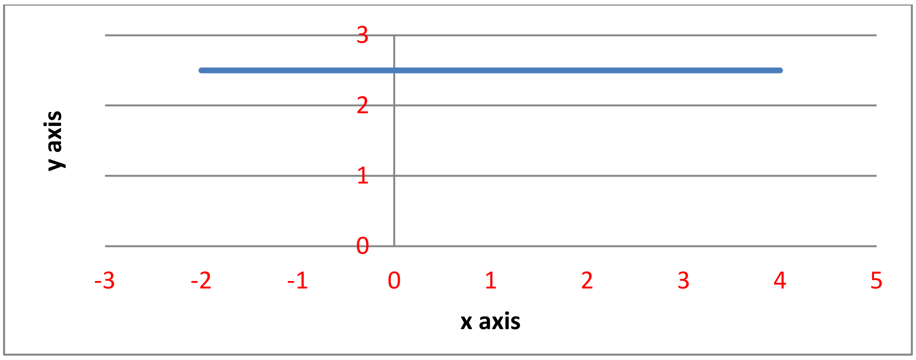
Or

Dom= R, set of all real numbers

Range= {k}

Graph of Constant Function

Let y = f(x) = k =2.5



The graph is a line parallel to axis of x

Identity Function

Y = f(x)=x, for all x

The rule here is: the value of y is always equals to x

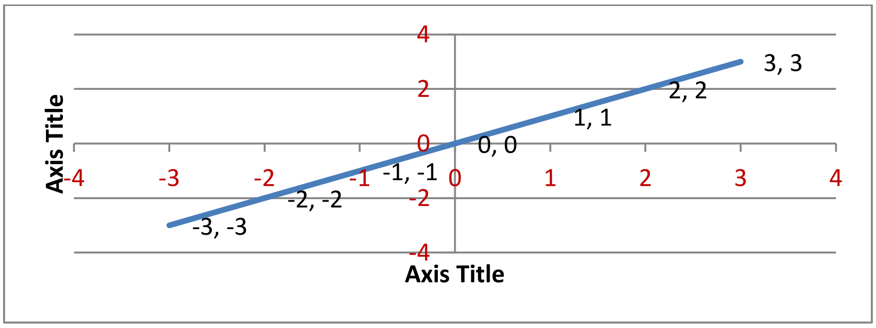
This is also a very simple rule in the sense that the value of y is identical with the value of x saving our time to calculate the value of y.

Dom = R

Range = R

i.e. Domain of the function is same as Range of the function

Graph of Identity Function



Modulus Function

$$y = f(x) = |x| = \begin{Bmatrix} x, x \geq 0 \\ -x, x < 0 \end{Bmatrix}$$

The rule here is: the value of y is always equals to the numerical value of x, not taking in to consideration the sign of x.

Example

Y =f(2)=2

Y=f(0)=0

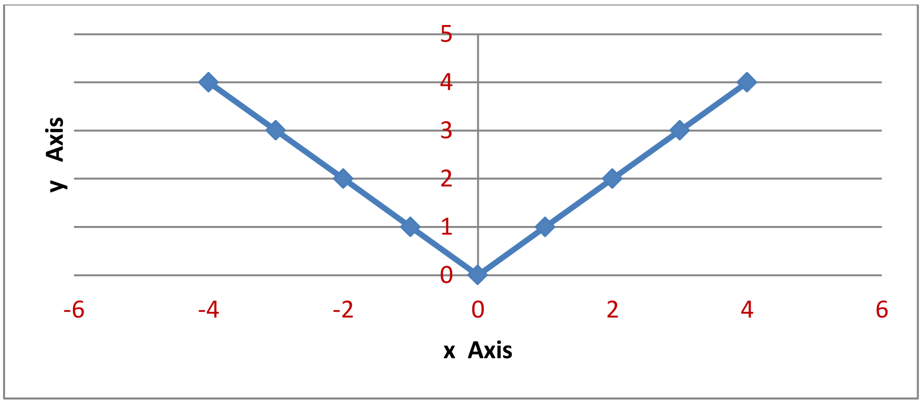
Y=f(-3)=3

This function is usually useful in dealing with values which are always positive for example, length, area etc.

Dom = R

Range = R^{+} U $\{0\}$

Graph of Modulus Function



Signum Function

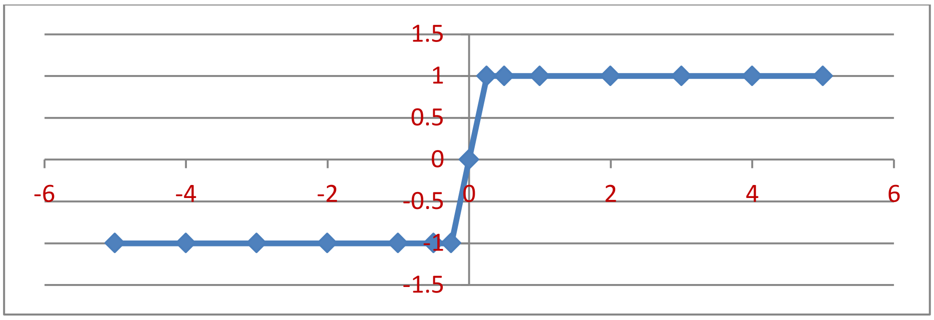
$$y = f(x) = \begin{Bmatrix} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{Bmatrix} = \begin{Bmatrix} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{Bmatrix}$$

This is also a very simple rule in the sense that the value of y is 1 if x is positive , 0 when x=0, and -1 when x is negative.

Dom = R

Range = $\{-1,0,1\}$

Graph of Signum Function



Greatest Integer Function

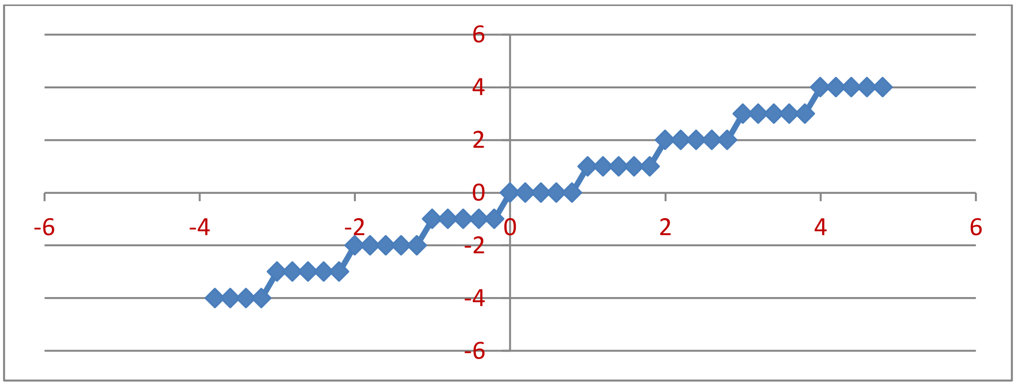
$$y = f(x) = [x] = greatest\ integer \leq x$$

For Example $[0] = 0, [0.2] = 0, [2.5] = 2, [-3.8] = -4\ , etc.$

Dom = R

Range=Z(set of all Integers)

Graph of The function



Exponential Function

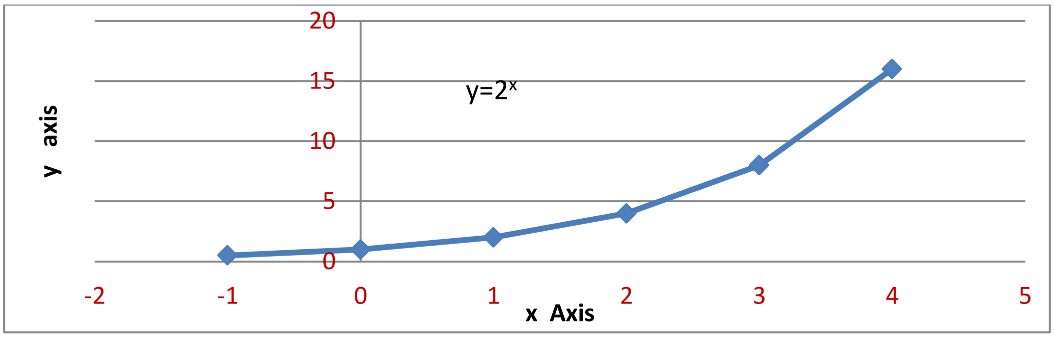
$$y = f(x) = a^x\ where\ a\ > 0$$

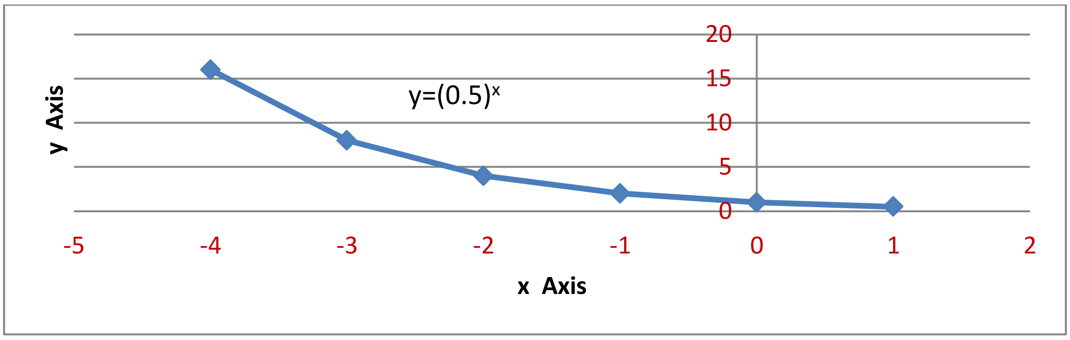
Dom = R

Range= R^+

The specialty of the function is that whatever the value of x, y can never be 0 or negative

Graph of Exponential Function





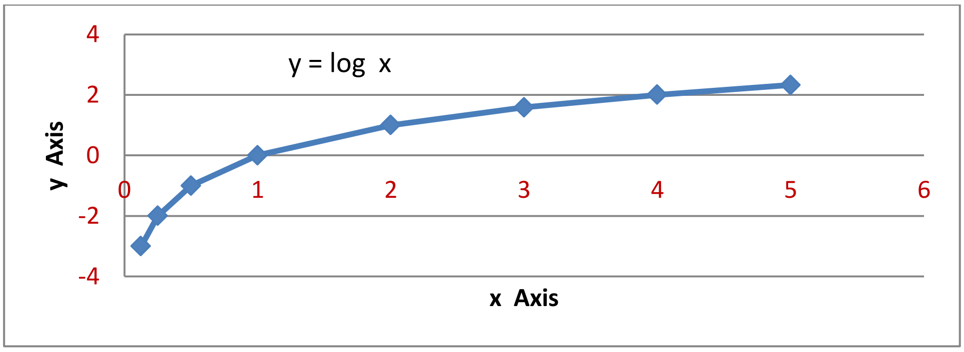
Logarithmic Function

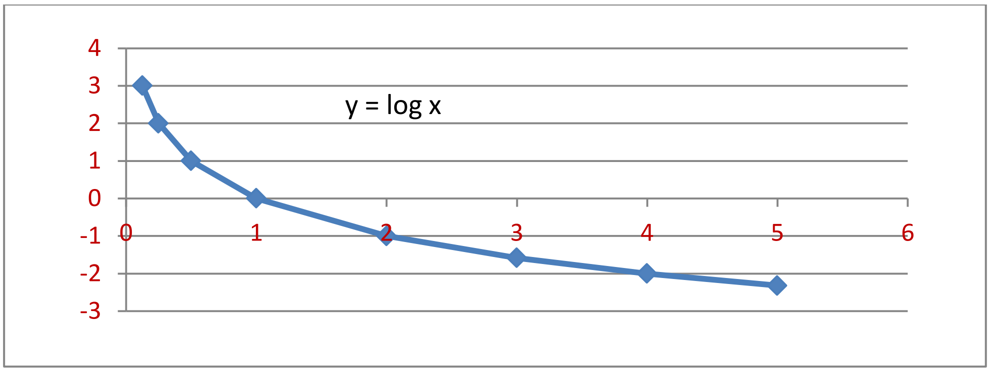
$$y = f(x) = \log_a x$$

Dom = R^+

Range =

Graph of Logarthmic Function




y = log x
4
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6

LIMIT OF A FUNCTION

Consider the function

$y = 2x + 1$

lets see what happens to value of y as the value of x changes.

Lets take the values of x close to the value of, say, 2. Now when we say value of x close 2. It can be a value like 2.1 or 1.9. in one case it is close to 2 but greater than 2 and in other it is close to 2 but less than 2.Now consider a sequence of such numbers slightly greater than 2 and slightly less than 2 and accordingly calculate the value of y in each case.

Look at the table

x	y=2x+1
1.9	4.8
1.91	4.82
1.92	4.84
1.93	4.86
1.94	4.88
1.95	4.9
1.96	4.92
1.97	4.94
1.98	4.96
1.99	4.98
2.01	5.02
2.02	5.04
2.03	5.06
2.04	5.08
2.05	5.1
2.06	5.12
2.07	5.14
2.08	5.16
2.09	5.18
2.1	5.2

We see in the tabulated value that

as x is approaching the value of 2 from either side, the value of y is approaching the value of 5

in other words we say,

y → 5 (y tends to 5) as x → 2(x tends to 2) or

$$\lim_{x \to 2} \quad y = 5$$

INFINITE LIMIT

As x→ a for some finite value of a, if the value of y is greater than any positive number however large then we say

Y → ∞ (y tends to infinity)

In other words y is said have an infinite limit as x → a. And we write

$$\lim_{x \to a} y = \infty$$

Example

If

$y = \frac{1}{x^2}$,

Then

$$\lim_{x \to 0} y = \infty$$

Since x→0, $x^2 \to 0$ *and* x^2 is positive,

$\frac{1}{x^2}$ becomes very very large and is positive. Therefore the result.

Similarly,

As x→ a for some finite value of a, if the value of y is less than any negative number however large then we say

Y →- ∞ (y tends to minus infinity)

In other words y is said have an infinite limit as x → a. And we write

$$\lim_{x \to a} y \ = -\infty$$

Example

If

$y = -\frac{1}{x^2}$,

Then

$$\lim_{x \to 0} y \ = -\infty$$

Since x→0, $x^2 \to 0 \; and \; x^2$ is positive,

$-\frac{1}{x^2}$ becomes very very large and is negative. Therefore the result.

LIMIT AT INFINITY

As x becomes very very large or in other words the value of x is greater than a very large positive number , i.e. x → ∞, if value of y is close to a finite value' a', then we say has a finite limit 'a' at infinity and write

$$\lim_{x \to \infty} y = a$$

Example

Let $y = \frac{1}{x}$

As x→∞ , $\frac{1}{x}$ becomes very very small and approaches the value 0.Therefore we write

$$\lim_{x \to \infty} y = 0$$

similarly

As x becomes very very large with a negative sign or in other words the value of x is less than a very large negative number , i.e. x →- ∞, if value of y is close to a finite value' a', then we say has a finite limit 'a' at infinity and write

$$\lim_{x\to-\infty} y = a$$

Example

Let $y = \frac{1}{x}$

As x→∞ , $\frac{1}{x}$ becomes very very small and approaches the value 0.Therefore we write

$$\lim_{x\to-\infty} y = 0$$

ALGEBRA OF LIMITS

1. Limit of sum of two functions is sum of their individual limits

Let $\lim_{x\to a} f(x) = m \; and \; let \; \lim_{x\to a} g(x) = n$, then

$$\lim_{x\to a}(f(x) + g(x)) = m + n$$

2.Limit of product of two functions is product of the their individual limits

Let $\lim_{x\to a} f(x) = m \; and \; let \; \lim_{x\to a} g(x) = n$, then

$$\lim_{x\to a}(f(x) \times g(x)) = m \times n$$

3. Limit of quotient of two functions is quotient of the their individual limits

Let $\lim_{x\to a} f(x) = m \; and \; let \; \lim_{x\to a} g(x) = n \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{m}{n}$$

SOME STANDARD LIMITS

1. $\lim_{x\to a} \; P(x) = P(a)$ where P(x) is polynomial in x

Example

$$\lim_{x\to 1}(2x^2 + 3x + 1) = 2 \times 1^2 + 3 \times 1 + 1 = 6$$

2. $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ where n is a rational number

Example

$$\lim_{x\to a} \frac{x^2 - a^2}{x - a} = 2a^{2-1} = 2a$$

3. $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e,$

$\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k,$

4. $\lim_{n\to 0}(1 + n)^{\frac{1}{n}} = e$

$\lim_{n\to 0}(1 + n)^{\frac{k}{n}} = e^k$

5. $\lim_{x\to 0} \left(\frac{a^x - 1}{x}\right) \quad = \ln a$

Example

.$\lim_{x\to 0} \left(\frac{2^x - 1}{x}\right) \quad = \ln 2$

6. $\lim_{x\to 0} \frac{log_a(1+x)}{x} = log_a \; e$

Example

$\lim_{x\to 0} \frac{ln(1+x)}{x} = \ln e = 1$

SOME STANDARD TRIGONOMETRIC LIMITS

1$\lim_{x\to 0}\sin x = 0$

2. $\lim_{x\to 0}\cos x = 1$

3. $\lim_{x\to 0}\tan x = 0$

4. $\lim_{x\to 0}\frac{\sin x}{x} = 1,$ here $x \to 0$ *through radian values*

5. $\lim_{x\to 0}\frac{\sin x^0}{x} = \frac{\pi}{180}$

Example

$$\lim_{x\to 0}\frac{\sin\ mx}{\sin\ nx} = \frac{m}{n}$$

Since

$$\lim_{x\to 0}\frac{\sin\ mx}{\sin\ nx} = \lim_{x\to 0}\quad \frac{\sin mx}{mx}\times\frac{nx}{\sin nx}\times\frac{m}{n}$$

$$= 1\times 1\times\frac{m}{n} = \frac{m}{n}$$

$$\lim_{x\to\infty}\frac{2x^2+3x+5}{3x^2+2x+1} = \lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{5}{x^2}}{3+\frac{2}{x}+\frac{1}{x^2}} = \frac{2}{3}$$

Example

$$\lim_{x\to\infty}\frac{1+2+3\ldots\ldots\ldots..+n}{n^2}$$

$$= \lim_{x\to\infty}\frac{n(n+1)}{2\times n^2} = \lim_{x\to\infty}\frac{(n^2+n)}{2\times n^2} = \lim_{x\to\infty}\frac{(1+\frac{1}{n})}{2} = \frac{1}{2}$$

Example

$$\lim_{x\to 0}\frac{(1-\mathrm{cos x})}{\mathrm{x}^2}=\lim_{x\to 0}\quad\frac{2sin^2\frac{x}{2}}{\mathrm{x}^2}=\frac{sin^2x}{2\frac{\mathrm{x}^2}{4}}=\frac{1}{2}\left(\lim_{x\to 0}\frac{sin\frac{x}{2}}{\frac{x}{2}}\right)\times\left(\lim_{x\to 0}\frac{sin\frac{x}{2}}{\frac{x}{2}}\right)$$

$$=\frac{1}{2}\times 1\times 1=\frac{1}{2}$$

Existence of Limits

When we say x tends to ‘a’ or write x→a it can happen in two different ways

X can approach ‘a’ through values greater than ‘a’ i.e from right side of ‘a’ on the Number Line

Or

X can approach ‘a’ through values smaller than ‘a’ i.e from left side of ‘a’ on the Number Line

The first case is called the Right Hand Limit and the later case is called the Left Hand Limit.

We, therefore conclude that Limit will exist iff the right Hand Limit and the Left Hand Limit both exist and are EQUAL

Consider the Greatest Integer Function

$$y=f(x)=[x]$$

Consider the limit of this function as $x\to 1$

The right hand limit of this function

$$\lim_{x\to 1+}[x]=1$$

Since if the value of x is greater than 1 for example 1+h,h>0, then the greatest integer less than equal to 1+h is 1

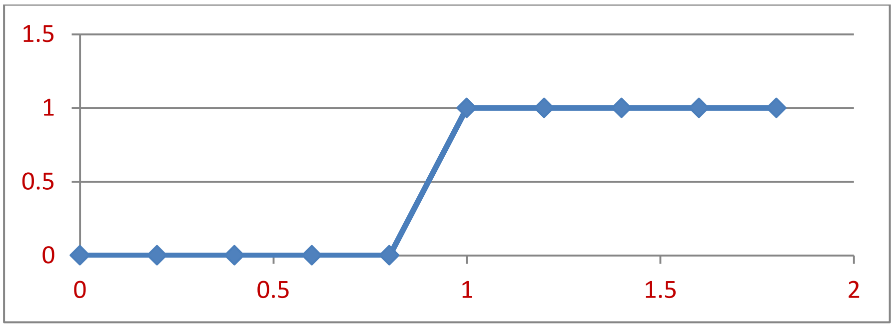
The left hand limit of this function

$$\lim_{x\to 1-}[x]=0$$

Since if the value of x is less than 1 for example 1-h,h>0, then the greatest integer less than equal to 1-h is 0

In this case the right hand limit and the left hand limit are not equal

And therefore the limit of this function as $x \to 1$does not exist



For that matter this function does not allow limit as $x \to n$

Since the right hand limit will be always n and the left hand limit will be n-1.

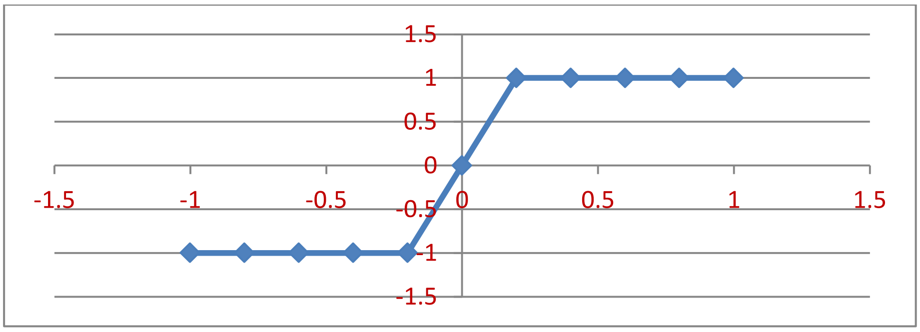
Consider the Signum Function

$$y = f(x) = \begin{Bmatrix} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{Bmatrix} = \begin{Bmatrix} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{Bmatrix}$$

Consider the limit of this function as $x \to 0$

The right hand limit of this function is 1 and the left hand limit of this function is -1 as evident from the definition of the function and concept of right and left hand limits

Therefore this function does not have a limit as $x \to 0$



Continuity of function

A function is continuous at a point ‘c’ iff its functional value i.e the value of the function at the point ‘c’ is same as limiting value of the function i.e value of the limit evaluated at the point ‘c’

OR

$$\lim_{x\to c} f(x) = f(c)$$

This means that a function is continuous at a point ‘c’ iff

All the three conditions mentioned below holds good

1.limit of the function as $x \to c$ exists

2.the function has a value at x=c. i.e f(c) does exist

3.the limit of the function is equal to value of the function at the point x=c

Most of the functions we encounter are continuous functions

For example

The physical growth of a child is a continuous function

The distance travelled is a continuous function of time

Continuous functions are easy to handle in the sense that we can predict the value at an latter stage. For example if the education of a child is continuous we can predict what he or she might be reading after say 5 years.

Examples

The constant function is continuous at any point ‘c’ and hence is continuous everywhere.

$$\lim_{x\to c} f(x) = \lim_{x\to c} K = K\,, where\ f(x) = \ K\ is\ the\ constant\ function$$

Consider the Function

$$f(x) = \frac{x^2 - 16}{x - 4}$$

This function is not continuous at x=4.Since the function is not defined at x=4

Consider another Function

$$f(x) = [x] \; or \; the \; greatest \; Integer \; Function$$

Consider the point x=2

This function does not have limit x → 2 as the Right Hand limit will be 2 and the Left Hand Limit will be 1.Hence this function is also not continuous at x = 2

Example

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4}, x \neq 4 \\ 8, x = 4 \end{cases}$$

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4) = 8 = f(4)$$

i.e

$$\lim_{x \to 4} f(x) = f(4)$$

This function is therefore continuous at x=4

Limiting value is same as functional value

Consider another Function

$$f(x) = \begin{cases} \left(1 + \dfrac{k}{x}\right)^{x}, x \neq 0 \\ e^k, x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(1 + \frac{k}{x}\right)^{x} = \lim_{x \to 0} \left[\left(1 + \frac{k}{x}\right)^{\frac{x}{k}}\right]^{k} = e^k$$

$$\lim_{x \to 0} f(x) = e^k = f(0)$$

i.e

limit of the function is same as value of the function at the point therefore, the function is continuous at x=0

example

consider the function

$$y = f(x) = \begin{cases} xsin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider the point x=0

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} xsin\frac{1}{x} = 0 = f(0)$$

Therefore the function is continuous at x=0

As,

$$0 \leq \left|xsin\frac{1}{x}\right| \leq |x|$$

Taking limit as x→ 0, we can conclude that

$$\lim_{x\to 0} xsin\frac{1}{x} = 0$$

Differentiation

A function f(x) is said to be differentiable at a point x=c iff

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \qquad exists$$

In general, a function is differentiable iff

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad exists$$

Once this limit exists, it is called the differential coefficient of f(x) or the derivative of the function f(x) at x=c

Or

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$$

Where $f'(c)$ and $f'(x)$ are the differential coefficient or the derivative of the function, the first being defined at x=c

Examples

Consider the function

$$y=f(x)=k\ or\ the\ constant\ function$$

In this case the differential coefficient $f'(x)$is given by

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{k-k}{\delta x}=0$$

Therefore the constant function is differentiable everywhere and the derivative is zero

Consider the function

$$y = f(x) = x^2$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{(x+\delta x) - x}$$

$$= 2x$$

Consider the function

$$y = f(x) = \sin x$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{sin(x+\delta x) \quad - sinx}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2cos\left(\frac{x+\delta x+x}{2}\right) \quad \times sin\left(\frac{x+\delta x-x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{cos\left(\frac{x+\delta x+x}{2}\right) \quad \times sin\left(\frac{x+\delta x-x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \frac{cos\left(\frac{x+\delta x+x}{2}\right) \quad \times sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \lim_{\delta x \to 0} cos\left(x + \frac{\delta x}{2}\right) \times 1$$

$$= \cos x$$

Therefore

$$y = f(x) = sinx$$

$$\frac{dy}{dx} = cosx$$

Algebra of derivatives

Consider two differentiable functions u(x) and v(x)

Let

$$y = u + v$$

Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Let

$$y = u \times v$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let

$$y = \frac{u}{v}, \qquad v \neq 0$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example

1

$$y = sinx + x^3$$

$$\frac{dy}{dx} = cosx + 2x$$

2

$$y = x^2cosx$$

$$\frac{dy}{dx} = x^2(-sinx) + cosx(2x)$$

$$= -x^2sinx + 2xcosx$$

3

$$y = \frac{sinx}{cosx}$$

$$\frac{dy}{dx} = \frac{cosxcosx - sinx(-sinx)}{(cosx)^2}$$

$$\frac{dy}{dx} = \frac{(cosx)^2 + (sinx)^2}{(cosx)^2}$$

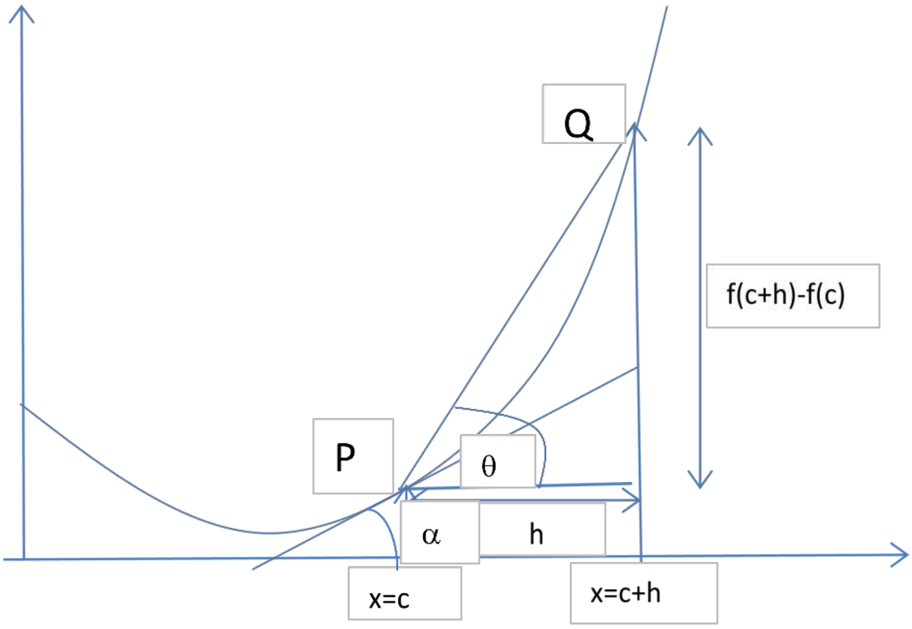
$$\frac{dy}{dx} = \frac{(cosx)^2 + (sinx)^2}{(cosx)^2}$$

$$\frac{dy}{dx} = \frac{1}{(cosx)^2} = (secx)^2$$

Geometrical meaning of $f'(c)$

Consider the graph of a function

$$y = f(x)$$



$$\frac{f(c+h)-f(c)}{h}$$

Represents the ratio of height to base of the angle the line joining the point P(c,f(c)) and Q (c+h,f(c+h))

i.e

$$\frac{f(c+h)-f(c)}{h}=tan\theta$$

Where θ is the angle the line joining the point P and Q makes with the positive direction of x axis.

In the limiting case as h→ 0 i.e as Q→ P the line PQ becomes the tangent line and the the angle $\theta\ becomes\ the\ angle\ \alpha$ which the tangent line makes with the positive direction of x axis

i.e

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)=tan\alpha=m\ (the\ slope\ of\ the\ tangent)$$

Application to Geometry

To find the equation of the tangent line to the curve y=f(x) at x=x_0

The equation of line passing through the point $(x_0, f(x_0))$ is give by

$$y-\ f(x_0)=m(x-x_0)$$

Where ‘ m’ is the slope of the tangent line.

As, we have seen

$$m=f'(x_0)$$

The equation is therefore

$$y-\ f(x_0)=f'(x_0)(x-x_0)$$

In the above example if we take

$f(x)=x^2\ and\ the\ point\ x_0=1$

The equation to the tangent at the point is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Or

$$y - 1^2 = 2 \times 1(x - 1)$$

where

$$f(x_0) = x_0^2 = 1^2 \; and \; f'(x_0) = 2 \times x_0 = 2 \times 1$$

i.e

the equation is

$$y - 1 = 2(x - 1)$$

Derivative as rate measurer

Remember the definition

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

The quantity

$$\frac{f(c+h) - f(c)}{h}$$

$$measures\ the\ rate\ of\ change\ in\ f(c) wth\ respect\ to\ change\ h\ in\quad 'c'$$

Consider the linear motion of a particle given as

$$s = f(t)$$

Where ‘s’ denotes the distance traversed and ‘t’ denotes the time taken

The ratio

$$\frac{s}{t}$$

Denotes the **average velocity** of the particle

To calculate the local velocity or instantaneous velocity at a point of time t=$t_0$we proceed in the following way

Consider an infinitesimal distance $'\ \delta s'$ traversed from time t=t_0 in time $'\ \delta t'$

The ratio

$$\frac{\delta s}{\delta t}$$

Still represents a average value of the velocity

The instantaneous velocity at t=t_0 can be calculated by considering the following limit

$$\lim_{\delta t \to 0} \frac{\delta s}{\delta t}$$

or

$$v\ = \frac{ds}{dt}$$

Where ‘v’ represents the instantaneous ***velocity*** which is defined as rate of change of displacement

Similarly, we can write the mathematical expression for ***acceleration***

As

$$a\ = \frac{dv}{dt}$$

Or the rate of change of velocity

Example

If the motion of a particle is given by

$$s = f(t) = 2t + 5$$

Which is linear in nature, we can calculate velocity at t=3

$$v(t = 3) = \frac{ds}{dt} = 2$$

It is clear that the velocity is independent of time ‘t’.

i.e

the above motion has constant or uniform velocity.

And, therefore, the acceleration

$$a = \frac{dv}{dt} = 0$$

Or the motion does not produce any acceleration.

Consider another motion of a particle given as

$$s = f(t) = 2t^2 + 3$$

Here the velocity at t=3 can be calculated as

$$v(t = 3) = \frac{ds}{dt} = 4t = 4 \times 3 = 12$$

And the acceleration

$$a = \frac{dv}{dt} = 4$$

Therefore we can say that the motion is said to have constant or uniform acceleration

Derivatives of implicit function

Consider the equation of a circle

$$x^2 + y^2 = r^2$$

This is an implicit function

Lets differentiate this equation with respect x throughout, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Derivative of parametric function

The equation of a circle can also be written as

$$x = rcost$$

$$y = rsint$$

This is called parametric function having parameter ‘t’

In this case

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{rcost}{-rsint} = \frac{x}{-y} = \frac{-x}{y}$$

Derivative of function with respect to another function

Consider the functions

$$y = f(x)$$

$$z = g(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{g'(x)}$$

Example

Let

$$y = sin(x)$$

$$z = x^3$$

$$\frac{dy}{dx} = \frac{f'(x)}{g'(x)} = \frac{cosx}{3x^2}$$

Derivative of composite function

Consider the function

$$y = f(u)\ where\ u = g(x)$$

Then y is called a composite function

In this case

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is called Chain Rule. This can be extended to any number of functions.

Example

1.Let

$$y = sinx^2$$

This can be written as

$$y = sinu$$

And

$$u = x^2$$

Applying chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = cosu \times 2x = 2xcosx^2$$

2.Let

$$y = tane^{x^2}$$

This can be written as

$$y = tanu$$

And

$$u = e^v$$

$$v = x^2$$

Applying chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = sec^2u \times e^v \times 2x = sec^2e^{x^2} \times e^{x^2} \times 2x$$

Derivatives of inverse function

$$since \quad \frac{\delta x}{\delta y} = \frac{1}{\frac{\delta y}{\delta x}}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

As $\delta x \to 0, \delta y\ also \to 0$

Which follows from the fact that

$$y = f(x)\ being\ a\ differentiable\ function\ is\ a\ continuous\ function$$

And the condition of continuity guarantees the above fact.

Derivative of inverse trigonometric function

Let

$$y = sin^{-1}x$$

Where $\quad y\epsilon\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

This can be written as

$$x = siny$$

$$\frac{dx}{dy} = cosy$$

Or

$$\frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\mp\sqrt{(1 - sin^2y)}} = \frac{1}{\mp\sqrt{(1 - x^2)}} = \frac{1}{\sqrt{(1 - x^2)}}$$

Since cosy is positive in the domain

Let

$$y = cos^{-1}x$$

Where $y\epsilon(0, \pi)$

This can be written as

$$x = cosy$$

$$\frac{dx}{dy} = -siny$$

Or

$$\frac{dy}{dx} = \frac{-1}{siny} = \frac{-1}{\mp\surd(1 - cos^2 y)} = \frac{-1}{\mp\surd(1 - x^2)} = \frac{-1}{\surd(1 - x^2)}$$

Since sin y is positive in the domain

Let

$$y = sec^{-1}x$$

Where $y\epsilon\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

This can be written as

$$x = secy$$

$$\frac{dx}{dy} = secy \times tany$$

Or

$$\frac{dy}{dx} = \frac{1}{secy \times tany} = \frac{1}{x\surd(sec^2 y - 1)} = \frac{1}{x(\mp\sqrt{(x^2 - 1)})} = \frac{1}{|x|\surd(1 - x^2)}$$

Since $secy \times tany$ is positive in the domain

Let

$$y = cosec^{-1}x$$

Where $y\epsilon\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$

This can be written as

$$x = cosecy$$

$$\frac{dx}{dy} = -cosecy \times coty$$

Or

$$\frac{dy}{dx} = \frac{-1}{cosecy \times coty} = \frac{-1}{x\sqrt{}(cosec^2y - 1)} = \frac{-1}{x(\mp\sqrt{(x^2-1)})} = \frac{-1}{|x|\sqrt{}(1-x^2)}$$

Since $cosecy \times coty$ is positive in the domain

Let

$$y = tan^{-1}x$$

This can be written as

$$x = tany$$

$$\frac{dx}{dy} = sec^2y$$

Or

$$\frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1+tan^2y} = \frac{1}{1+x^2}$$

Let

$$y = cot^{-1}x$$

This can be written as

$$x = coty$$

$$\frac{dx}{dy} = -cosec^2y$$

Or

$$\frac{dy}{dx} = \frac{-1}{cosec^2 y} = \frac{-1}{1 + cot^2 y} = \frac{-1}{1 + x^2}$$

Higher order derivatives

Let

$$y = f(x)$$

Is differentiable and also

$$\frac{dy}{dx} = f'(x)$$

Is differentiable. Then we define

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2} = f''(x)$$

$$=$$

$$= \lim_{\delta x \to 0} \frac{f'(x + \delta x) - f'(x)}{\delta x}$$

This is the 2nd. Order derivative of the function

Similarly we can define higher order derivatives of the function

Example

Let

$$y = f(x) = x^3 + x^2 + x + 1$$

$$\frac{dy}{dx} = f'(x) = 3x^2 + 2x + 1$$

$$\frac{d^2 y}{dx^2} = f''(x) = 6x + 2$$

Consider the Function

$$y = f(x) = Acosx + Bsinx$$

Here

$$\frac{dy}{dx} = f'(x) = -Asinx + Bcosx$$

$$\frac{d^2y}{dx^2} = f''(x) = -Acosx - Bsinx = -y$$

i.e in this case

$$\frac{d^2y}{dx^2} + y = 0$$

Monotonic Function

Increasing function

Consider a function

$$y = f(x)$$

If $x_2 > x_1 \ \ implies\ f(x_2) > f(x_1)$

Then the function is increasing

Example

$$y = f(x) = x + 1$$

$$f(2) = 2 + 1 = 3$$

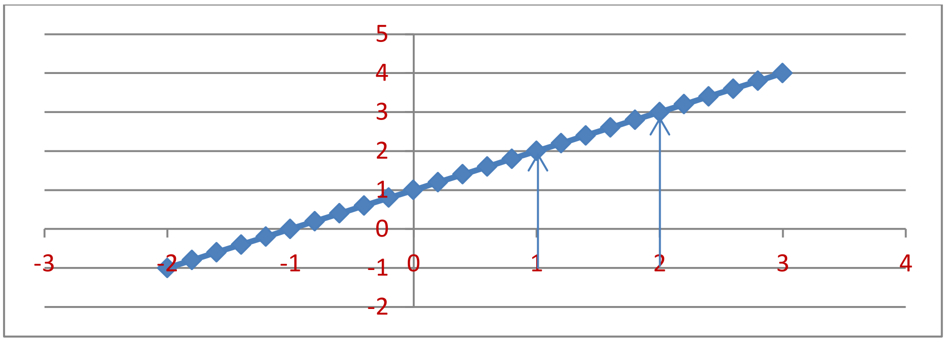
$$f(1) = 1 + 1 = 2$$

Or

$$f(2) > f(1)$$

Therefore the function is increasing

Graph of the function



Decreasing function

Consider a function

$$y = f(x)$$

If $x_2 > x_1 \ implies\ f(x_2) < f(x_1)$

Then the function is decreasing

Consider the function

$$y = f(x) = \frac{1}{x}$$

$$f(2) = \frac{1}{2}$$

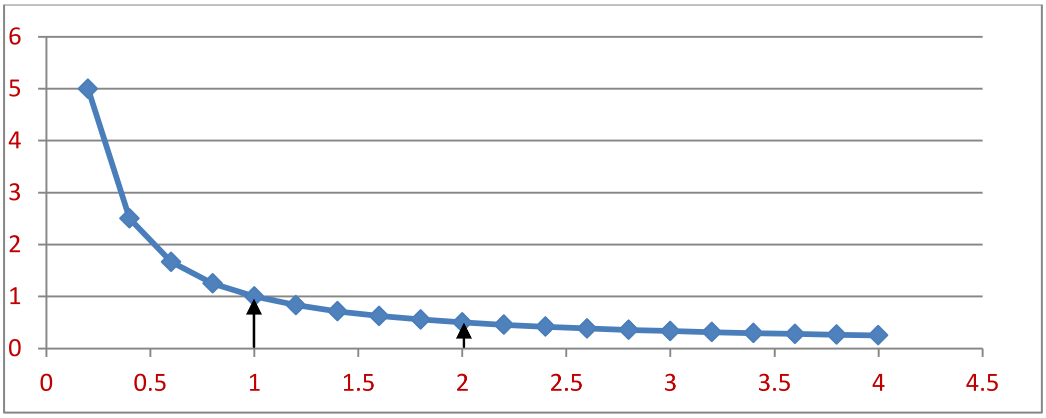
$$f(1) = \frac{1}{1} = 1$$

Or

$$f(2) < f(1)$$

Therefore the function is decreasing

Graph of the function



A function either increasing or decreasing is called monotonic.

Derivative of Increasing Function

If f(x) is increasing, then

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} > 0$$

i.e

for increasing function the derivative is always positive

Derivative of Decreasing Function

If f(x) is decreasing, then

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} < 0$$

i.e

for decreasing function the derivative is always negative

example

let

$$y = f(x) = x + 1$$

$$\frac{dy}{dx} = f'(x) = 1 > 0$$

Therefore the function is increasing

Let

$$y = f(x) = \frac{1}{x}$$

$$\frac{dy}{dx} = f'(x) = \frac{-1}{x^2} < 0$$

Therefore the function is decreasing

Let

$$y = f(x) = x^2$$

$$\frac{dy}{dx} = f'(x) = 2x$$

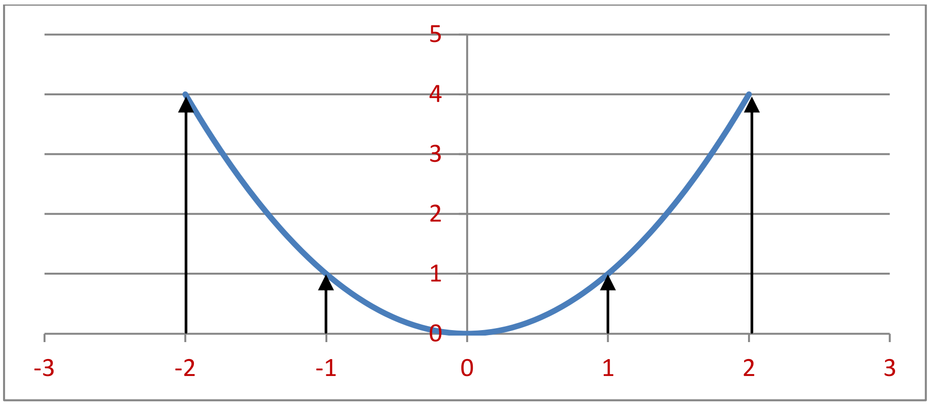
$$> 0 \; for \; x > 0$$

$$< 0 \; for \; x < 0$$

Therefore the function is increasing for $x > 0$ and decreasing for $x < 0$

Graph of the function

$$y = f(x) = x^2$$



MAXIMA AND MINIMA OF A FUNCTION

Consider a function

$$y = f(x)$$

Consider the point x=c

If at this point

$$f(c) > f(c+h), where\ |h| < \delta$$

Then f(c) is called local maximum or simply a maximum of the function

If at this point

$$f(c) < f(c+h), where\ |h| < \delta$$

Then f(c) is called a local minimum or simply a minimum

A function can have several local maximum values and several local minimum values in its domain and it is possible that a local minimum can be larger than a local maximum.

If f (c) is a local maximum then the graph of the function in the domain

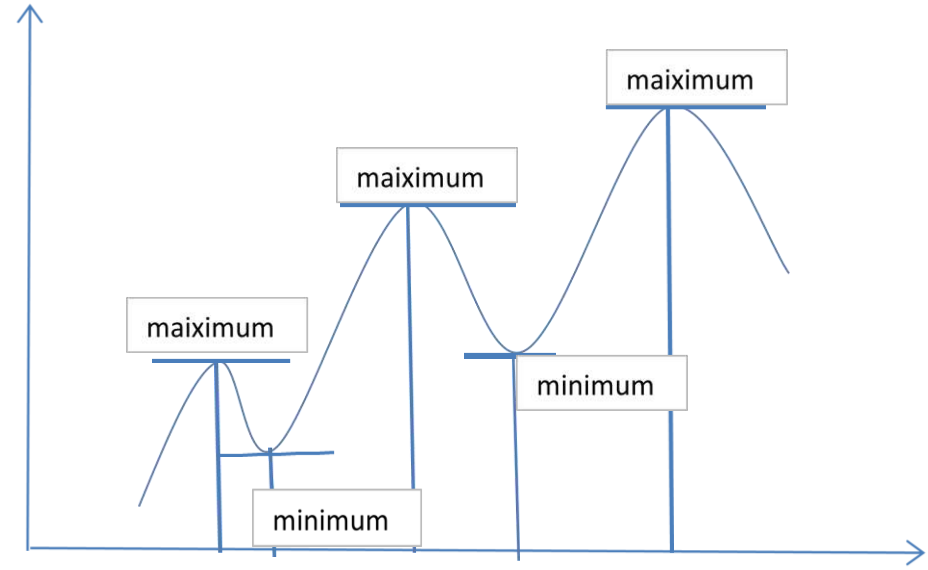
$$(c - \delta, c + \delta)$$

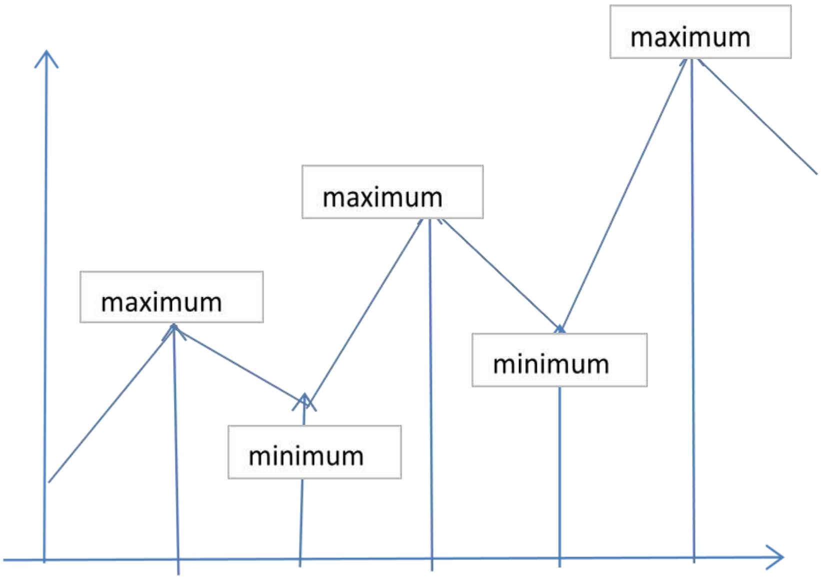
Will be concave downwards

If f (c) is a local minimum then the graph of the function in the domain

$$(c - \delta, c + \delta)$$

Will be concave upwards

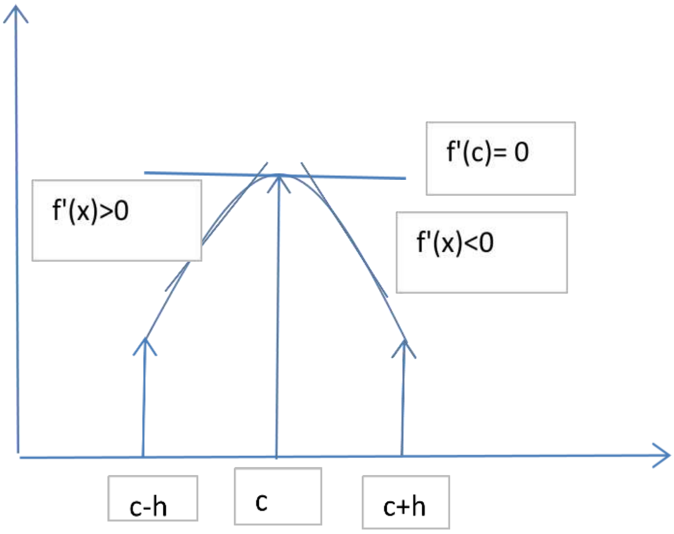


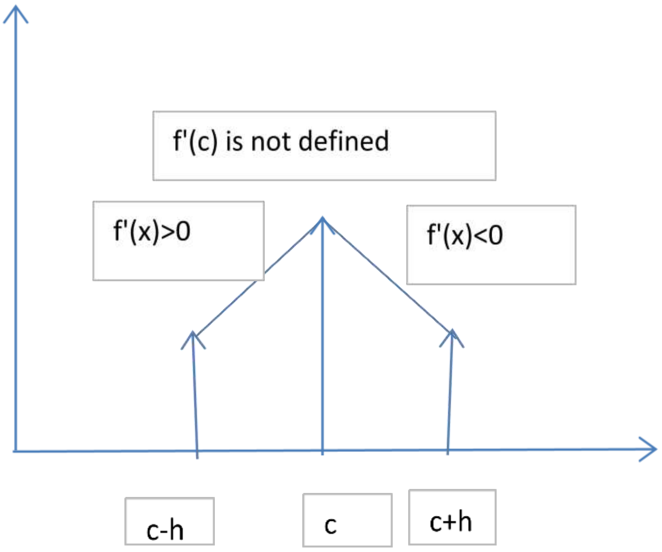


Maximum Case

In other words at a point of local maximum the function is increasing on the left of the point and decreasing on the right of the point

Therefore the derivative of the function changes sign from positive to negative as it passes through x=c



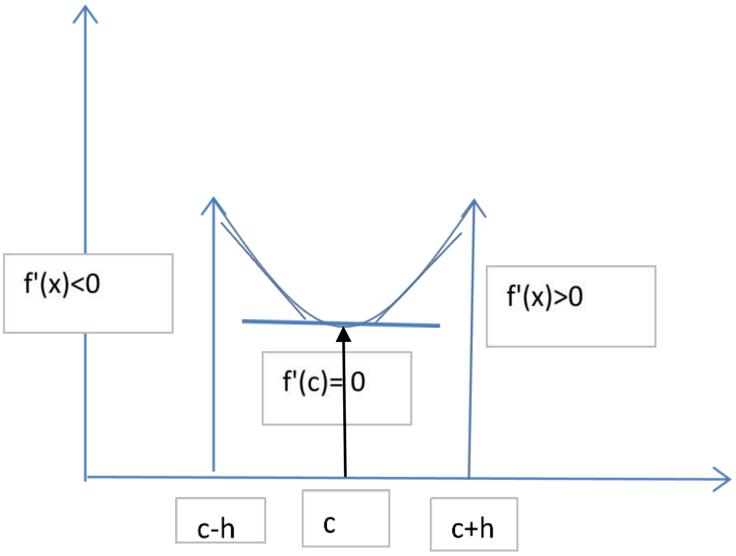


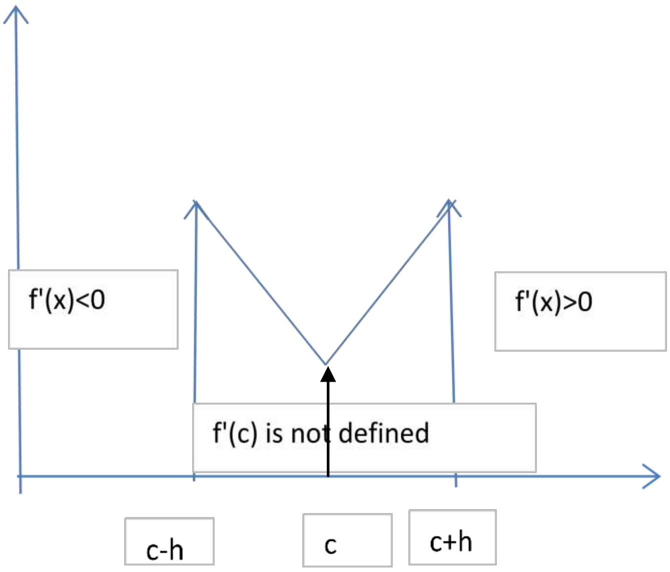
Therefore we conclude that the derivative of the function is a decreasing function and as such its derivative i.e the second order derivative is negative

Minimum Case

At a point of local minimum the function is decreasing on the left of the point and increasing on the right of the point

Therefore the derivative of the function changes sign from negative to positive as it passes through x=c





Therefore we conclude that the derivative of the function is a increasing function and as such its derivative i.e the second order derivative is positive

In either maximum or minimum case the 1st. derivative of the function is zero or is not defined at the point of maximum or minimum

The point x=c where the derivative vanishes or does not exist at all is called a critical point or turning point or stationary point.

A function can have neither a maximum nor a minimum value

Example

Consider the function

$$y = f(x) = x^3$$

Here

$$\frac{dy}{dx} = 3x^2$$

This vanishes at x=0

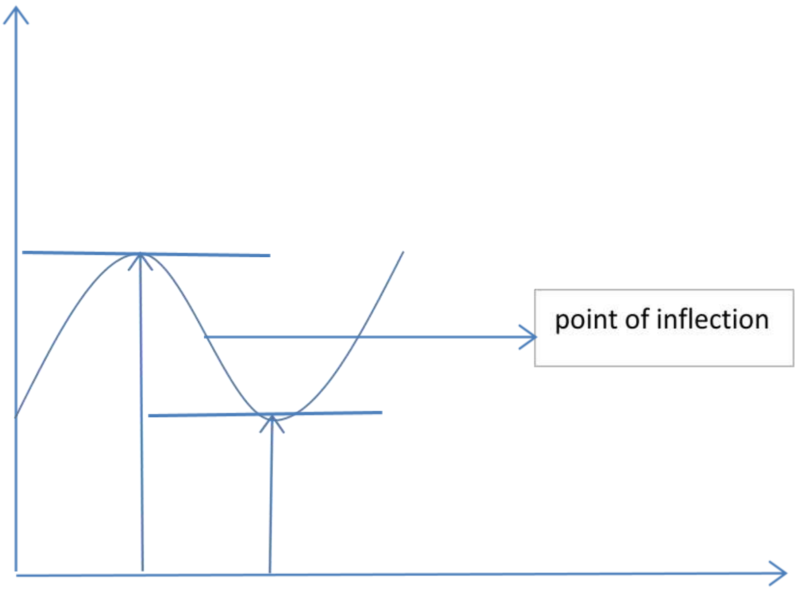
And

$$\frac{d^2y}{dx^2} = 6x$$

Which also vanishes

Therefore we may conclude that the function does not have maximum neither minimum value

Point of inflexion



If a curve is changing its nature from concave downwards to concave upwards as shown in the figure or vice versa, then at the point where this change occurs is called the point of inflexion. In other words on one side of the point of inflexion the curve is concave downward and on the other side the curve is concave upward or vice versa

In the above figure,

On the left side of point of inflexion a maximum value occurs and to the right side of point of inflexion a minimum value occurs.

In other words, remembering the condition of maximum and minimum, we can say,

The 2nd order derivative changes its sign from negative to positive as in the case given in the figure or vice versa.

In other words the point of inflexion is the point of either maximum or minimum of the 1st derivative of the function

Hence at the point of inflexion the 2nd. order derivative vanishes or is not defined and the 2nd. order derivative changes its sign as it passes through the point of inflexion i.e at the point of inflexion

1. $\frac{d^2y}{dx^2} = f''(x) = 0$ or is not defined
2. The 2nd.order derivative changes sign as it passes through the point

Example

Consider the function we discussed earleier

$$y = f(x) = x^3$$

Here

$$\frac{dy}{dx} = 3x^2$$

This vanishes at x=0

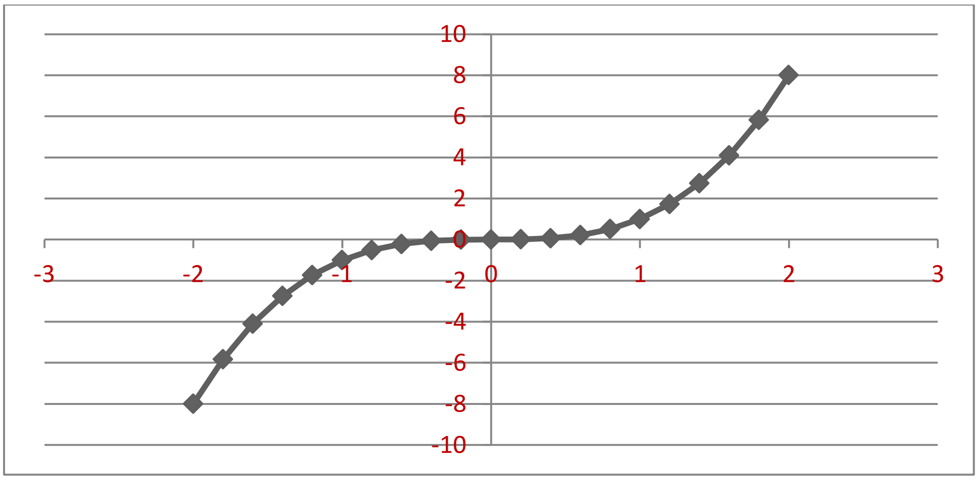
And

$$\frac{d^2y}{dx^2} = 6x$$

Which also vanishes at x=0

But

$$\frac{d^3y}{dx^3} = 6 \neq 0$$



Therefore we conclude that

X=0 is a point of inflexion for the curve

Working procedure to find the maxima and minima

1.Given any function, equate the first derivative to zero to find the turning points or critical points

2.Test the sign of the second derivative at these points. If the sign is negative it is appoint of maximum value. If the sign is positive it is a point of minimum value.

3. then calculate the maximum value/minimum value of the function by taking the value of x as the point

Example

If the sum of two numbers is 10, find the numbers when their product is maximum

Solution

Let the numbers be x and 10-x

Let

$$y = f(x) = x(10 - x)$$

$$= 10x - x^2$$

$$\frac{dy}{dx} = 10 - 2x = 0$$

$$x = 5$$

$$\frac{d^2y}{dx^2} = -2 < 0$$

Therefore the function which is the product of the numbers maximum if the numbers are equal i.e 5 and 5.

EXAMPLE

Investigate the extreme values of the function

$$f(x) = x^4 - 2x^2 + 3$$

The critical points are roots of the equation

$$f'(x) = 4x^3 - 4x \quad = 0$$

Or

$$f'(x) = 4x(x^2 - 1) = 0$$

Or

$$x = 0, x = 1, x = -1$$

Lets check the sign of the 2nd. Derivative at these points

Now,

$$f''(x) = 4(3x^2 - 1)$$

$$f''(0) = 4(-1) = -4 < 0$$

Therefore x=0 is a point of maximum value.

The maximum value of the function is given as

$$f(x)_{max} = f(0) = 3$$

Now

$$f''(1) = 4(3 - 1) = 8 > 0$$

Therefore x = 1 is a point of minimum value.

The minimum value of the function is given as

$$f(x)_{min} = f(1) = 1 - 2 + 3 = 2$$

Now

$$f''(-1) = 4(3 - 1) = 8 > 0$$

Therefore x = -1 is a point of minimum value.

The minimum value of the function is given as

$$f(x)_{min} = f(-1) = 1 - 2 + 3 = 2$$

INTEGRATION AS INVERSE PROCESS OF DIFFERENTIATION

Integration is the process of inverse differentiation .The branch of calculus which studies about Integration and its applications is called Integral Calculus.

Let $F(x)$ and $f(x)$ be two real valued functions of x such that,

$$\frac{d}{dx}F(x) = f(x)$$

Then, $F(x)$ is said to be an anti-derivative (or integral) of $f(x)$.
Symbolically we write $\int f(x)\,dx = F(x)$.

The symbol $\int$ denotes the operation of integration and called the integral sign.
$'dx'$ denotes the fact that the Integration is to be performed with respect to x .The function $f(x)$ is called the Integrand.

INDEFINITE INTEGRAL

Let $F(x)$be an anti-derivative of $f(x)$.
Then, for any constant 'C',

$$\frac{d}{dx}\{F(x)+C\} = \frac{d}{dx}F(x) = f(x)$$

So, $F(x)+C$ is also an anti-derivative of $f(x)$, where C is any arbitrary constant. Then, $F(x)+C$ denotes the family of all anti-derivatives of $f(x)$, where C is an indefinite constant.

Therefore, $F(x)+C$ is called the Indefinite Integral of $f(x)$.
Symbolically we write

$$\int f(x)\,dx = F(x) + C,$$

Where the constant C is called the constant of integration. The function $f(x)$ is called the Integrand.

Example :-Evaluate $\int \cos x\ dx$.
Solution:-We know that

$$\frac{d}{dx}\sin x = \cos x$$

So, $$\int \cos x\ dx = \sin x + C$$

ALGEBRA OF INTEGRALS

1. $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$

2. $\int k\,f(x)dx = k\int f(x)dx,$ for any constant k.

3. $\int [a\,f(x) + b\,g(x)]\,dx = a\int f(x)dx + b\int g(x)dx,$
for any constant a & b

INTEGRATION OF STANDARD FUNCTIONS

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ \ ,(n \neq -1)$
2. $\int \frac{1}{x} dx = \ln|x| + C$
3. $\int cosx\, dx = \sin x + C$
4. $\int \sin x\, dx = -\cos x + C$
5. $\int sec^2 x\, dx = \tan x + C$
6. $\int cosec^2 x\, dx = -\cot x + C$
7. $\int \sec x \tan x\, dx = \sec x + C$
8. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
9. $\int e^x dx = e^x + C$
10. $\int a^x dx = \frac{a^x}{\ln a} + C \ \ ,(a > 0)$
11. $\int \tan x\, dx = \ln|\sec x| + C = -\ln|\cos x| + C$
12. $\int \cot x\, dx = \ln|\sin x| + C = -\ln|\operatorname{cosec} x| + C$
13. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
14. $\int \operatorname{cosec} x\, dx = \ln|\operatorname{cosec} x - \cot x| + C$
15. $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + C$
16. $\int \frac{1}{1+x^2} dx = tan^{-1}x + C$
17. $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1}x + C$
18. $\int \frac{1}{\sqrt{x^2+1}} dx = \ln\left|x + \sqrt{x^2+1}\right| + C$
19. $\int \frac{1}{\sqrt{x^2-1}} dx = \ln\left|x + \sqrt{x^2-1}\right| + C$

Example:- Evaluate $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

Solution:- $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

$$= \int \frac{a^2 sin^2 x + b^2 cos^2 x}{4 sin^2 x . cos^2 x} dx$$

$$= \frac{a^2}{4} \int \frac{1}{cos^2 x} dx + \frac{b^2}{4} \int \frac{1}{sin^2 x} dx$$

$$= \frac{a^2}{4} \int sec^2 x\, dx + \frac{b^2}{4} \int cosec^2 x\, dx$$

$$= \frac{1}{4} [a^2 \tan x - b^2 \cot x] + C$$

INTEGRATION BY SUBSTITUTION

When the integrand is not in a standard form, it can sometimes be transformed to integrable form by a suitable substitution.

The integral $\int f\{g(x)\} g'(x) dx$ can be converted to
$\int f(t) dt$ by substituting $g(x)$ by t.

So that, if $\int f(t)dt = F(t) + C$,then

$\int f\{g(x)\}g'(x)dx = F\{g(x)\} + C.$

This is a direct consequence of CHAIN RULE.
For,

$$\frac{d}{dx}[F\{g(x)\} + C] = \frac{d}{dt}[F(t) + C].\frac{dt}{dx} = f(t).\frac{dt}{dx} = f\{g(x)\}g'(x)$$

There is no fixed formula for substitution.

Example:- Evaluate $\int \cos(2 - 7x)\, dx$
Solution:- Put $t = 2 - 7x$
So that $\frac{dt}{dx} = -7 \Rightarrow dt = -7dx$

$$\therefore \quad \int \cos(2 - 7x)\, dx = \frac{-1}{7}\int \cos t\, dt$$
$$= \frac{-1}{7}\sin t + C$$
$$= \frac{-1}{7}\sin(2 - 7x) + C$$

INTEGRATION BY DECOMPOSITION OF INTEGRAND

If the integrand is of the form $\sin mx \cdot \cos nx$, $\cos mx \cdot \cos nx$ or $\sin mx \cdot \sin nx$, then we can decompose it as follows;

1. $\sin mx \cdot \cos nx = \frac{1}{2} \cdot 2 \sin mx \cdot \cos nx = \frac{1}{2}[\sin(m + n)x + \sin(m - n)x]$
2. $\cos mx \cdot \cos nx = \frac{1}{2}[\cos(m - n)x + \cos(m + n)x]$
3. $\sin mx \cdot \sin nx = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x]$

Similarly, in many cases the integrand can be decomposed into simpler form, which can be easily integrated.
Example:-Integrate $\int \sin 5x \cdot \cos 2x\, dx$

Solution:- $\int \sin 5x \cdot \cos 2x\, dx$

$$= \frac{1}{2}\int[\sin(5 + 2)x + \sin(5 - 2)x]\, dx$$
$$= \frac{1}{2}\int(\sin 7x + \sin 3x)\, dx$$
$$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x - \frac{1}{3}\cos 3x\right] + C$$
$$= -\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + C$$

Example:-Integrate $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx$

Solution:- $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx = \int \frac{2\sin 5x \cos x}{2\cos 5x \cos x}dx$

$$= \int \frac{\sin 5x}{\cos 5x}dx$$

Put t = cos 5x, so that $\frac{dt}{dx} = -5\, sin5x \Rightarrow dt = -5sin5x.dx$

$\therefore \int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx = -\frac{1}{5}\int \frac{dt}{t} = -\frac{1}{5}ln|t| + C$

$$= -\frac{1}{5}\ln|\cos 5x| + C$$
$$= \frac{1}{5}\ln|\sec 5x| + C$$

INTEGRATION BY PARTS

This rule is used to integrate the product of two functions.
If u and v are two differentiable functions of x, then according to this rule have;

$$\int uv\, dx = u\int v\, dx - \int\left[\frac{du}{dx}\int v\, dx\right]dx$$

In words, Integral of the product of two functions

$$= \boldsymbol{first\ function} \times (\boldsymbol{Integral\ of\ second\ function}) - \boldsymbol{Integral\ of}(\boldsymbol{derivative\ of\ first} \times \boldsymbol{Integral\ of\ second})$$

The rule has been applied with a proper choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' functions .Usually from among exponential function(**E**),trigonometric function(**T**),algebraic function(**A**),Logarithmic function(**L**) and inverse trigonometric function(**I**),the choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' function is made in the order of **ILATE.**

Example:-Evaluate$\int x\sin x\, dx$

Solution:-$\int x\sin x\, dx$

$$= x\int \sin x\, dx - \int\left[\frac{dx}{dx}.\int \sin x\, dx\right]dx$$
$$= -x\cos x + \int \cos x\ dx$$
$$= \sin x - x\cos x + C$$

Example:-Evaluate$\int e^x\cos 2x\, dx$

Solution:- $\int e^x\cos 2x\, dx = e^x\cos 2x - \int e^x(-2\sin 2x)\, dx$

$$= e^x\cos 2x + 2\int e^x\sin 2x\, dx$$
$$= e^x\cos 2x + 2\left[e^x\sin 2x - 2\int e^x\cos 2x\, dx\right]$$
$$= e^x\cos 2x + 2\, e^x\sin 2x - 4\int e^x\cos 2x\, dx + K$$

So,5 $\int e^x\cos 2x = e^x[\cos 2x + 2\sin 2x] + K$

$\therefore \int e^x\cos 2x\, dx = \frac{e^x}{5}[\cos 2x + 2\sin 2x] + C$ (where = $K/2$)

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

The irrational forms $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$ can be simplified to radical free functions as integrand by putting $x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$ respectively.
The substitution $x = a\tan\theta$ can be used in case of presence of $x^2 + a^2$ in the integrand, particularly when it is present in the denominator.

ESTABLISHMENT OF STANDARD FORMULAE

1. $\int\frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + C$

5. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

Solutions:

1. Let $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$ and $\theta = sin^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2sin^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \int d\theta = \theta + C = sin^{-1}\frac{x}{a} + C$

2. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\ d\theta$ and $\theta = tan^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta\ d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\sec^2\theta\ d\theta}{a^2(\tan^2\theta+1)} = \int \frac{asec^2\theta}{a^2sec^2\theta} d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$

$= \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\ d\theta$ and $\theta = sec^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\ d\theta}{a\sec\theta\sqrt{a^2sec^2\theta-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\, a\tan\theta} d\theta = \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + C = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\ d\theta$.

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\ d\theta}{\sqrt{a^2tan^2\theta+a^2}} = \int \frac{asec^2\theta}{a\sec\theta} d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + K$

$= \ln\left|\sqrt{tan^2\theta+1} + \tan\theta\right| + K = \ln\left|\sqrt{\frac{x^2}{a^2}+1} + \frac{x}{a}\right| + K$

$= \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + K$

$= \ln\left|x+\sqrt{x^2+a^2}\right| + K - \ln|a|$

$= \ln\left|x+\sqrt{x^2+a^2}\right| + C$ (Where $C = K - \ln|a|$)

5. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\ d\theta$

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\ d\theta}{\sqrt{a^2sec^2\theta-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta} d\theta = \int \sec\theta\ d\theta$

$= \ln|\sec\theta + \tan\theta| + K = \ln\left|\sec\theta + \sqrt{sec^2\theta - 1}\right| + K$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + K$

$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + K$

$= \ln\left|x+\sqrt{x^2-a^2}\right| + K - \ln|a|$

$= \ln\left|x+\sqrt{x^2-a^2}\right| + C$ (Where $C = K - \ln|a|$)

SOME SPECIAL FORMULAE

1. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$

2. $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

3. $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

Solutions:

1.
$$\int \sqrt{a^2-x^2}dx = \int 1 \cdot \sqrt{a^2-x^2}dx$$
$$= x\sqrt{a^2-x^2} - \int x\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)dx$$
$$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$$
$$= x\sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}dx$$
$$= x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \sqrt{a^2-x^2}dx$$
$$\therefore \quad 2\int \sqrt{a^2-x^2}dx = x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$
$$= x\sqrt{a^2-x^2} + a^2 sin^{-1}\frac{x}{a} + K$$
$$\therefore \quad \int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\frac{x}{a} + C \qquad (\text{ Where } C = \frac{K}{2})$$

2.
$$\int \sqrt{x^2+a^2}dx = \int 1 \cdot \sqrt{x^2+a^2}dx$$
$$= x\sqrt{x^2+a^2} - \int x\left(\frac{2x}{2\sqrt{x^2+a^2}}\right)dx$$
$$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$$
$$= x\sqrt{x^2+a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx$$
$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}dx + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$$
$$\therefore \quad 2\int \sqrt{x^2+a^2}dx = x\sqrt{x^2+a^2} + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$$
$$\text{So, } 2\int \sqrt{x^2+a^2}dx = x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + K$$
$$\therefore \quad \int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$$
$$(\text{ Where } C = \frac{K}{2})$$

3.
$$\int \sqrt{x^2-a^2}dx = \int 1 \cdot \sqrt{x^2-a^2}dx$$
$$= x\sqrt{x^2-a^2} - \int x\left(\frac{2x}{2\sqrt{x^2-a^2}}\right)dx$$
$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$$
$$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}dx$$
$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}dx + a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$
$$\therefore \quad 2\int \sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$
$$\text{So, } 2\int \sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2 \ln\left|x+\sqrt{x^2-a^2}\right| + K$$
$$\therefore \quad \int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$
$$(\text{ Where } C = \frac{K}{2})$$

METHOD OF INTEGRATION BY PARTIAL FRACTIONS

If the integrand is a proper fraction $\frac{P(x)}{Q(x)}$, then it can be decomposed into simpler partial fractions and each partial fraction can be integrated separately by the methods outlined earlier.

SOME SPECIAL FORMULAE

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

Solutions:

1. We have, $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C$$

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

2. We have, $\frac{1}{a^2-x^2} = \frac{1}{(a+x)(a-x)}$

$$= \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)$$

$$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int\left(\frac{1}{a+x} + \frac{1}{a-x}\right)dx$$

$$= \frac{1}{2a}\left[\ln|a+x| - \ln|a-x|\right] + C$$

$$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

Example:- Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

Solution:- Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$ ---------------(1)

Multiplying both sides of (1) by $(x-1)^2(x+3)$,

$\Longrightarrow x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$ ---------------(2)

Putting $x = 1$ in (2),we get,$B = \frac{1}{2}$

Putting $x = -3$ in (2),we get,$10 = 16C \Longrightarrow C = \frac{5}{8}$

Equating the co-efficients of x^2 on both sides of (2), we get

$1 = A + C \Longrightarrow A = 1 - \frac{5}{8} = \frac{3}{8}$

Substituting the values of A,B &C in (1) ,we get

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}\cdot\frac{1}{x+3}$$

$$\therefore \int \frac{x^2+1}{(x-1)^2(x+3)}dx$$

$$= \frac{3}{8}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{(x-1)^2} + \frac{5}{8}\int\frac{dx}{x+3}$$

$$= \frac{3}{8}\ln|x-1| + \frac{5}{8}\ln|x+3| - \frac{1}{2(x-1)} + C$$

Example:- Evaluate $\int \frac{x}{(x-1)(x^2+4)}dx$

Solution:- Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ ---------------(1)
Multiplying both sides of (1) by $(x-1)(x^2+4)$,we get
$x = A(x^2+4) + (Bx+C)(x-1)$--------------(2)
Putting $x = 1$ in (2), we get,$A = \frac{1}{5}$
Putting $x = 0$ in (2), we get,$0 = 4A - C \Longrightarrow C = 4A \Longrightarrow C = \frac{4}{5}$
Equating the co-efficients of x^2 on both sides of (2), we get
$0 = A + B \Rightarrow B = -\frac{1}{5}$
Substituting the values of A, B and C in (1) we get

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

$\therefore \quad \int \frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{x-4}{x^2+4}dx$
$= \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$
$= \frac{1}{5}\int \frac{dx}{x-1} + \frac{1}{10}\int \frac{2xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$
$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| + \frac{2}{5}tan^{-1}\left(\frac{x}{2}\right) + C$

Example:- Evaluate $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$

Solution:- Let $x^2 = y$ $Then \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ ---------------(1)
Multiplying both sides of (1) by $(y+1)(y+4)$,we get
$y = A(y+4) + B(y+1)$-----------(2)
Putting $y = -1$ and $y = -4$ successively in (2),we get,$A = -\frac{1}{3}$ and$B = \frac{4}{3}$
Substituting the values of A and B in (1),we get

$$\frac{□}{(□+1)(□+4)} = -\frac{1}{3(□+1)} + \frac{4}{3(□+4)}$$

Replacing □ by $□^2$, we obtain

$$\frac{□^2}{(□^2+1)(□^2+4)} = -\frac{1}{3(□^2+1)} + \frac{4}{3(□^2+4)}$$

$\therefore \quad \int \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{-1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$
$= -\frac{1}{3}tan^{-1}x + \frac{2}{3}tan^{-1}\left(\frac{x}{2}\right) + C$

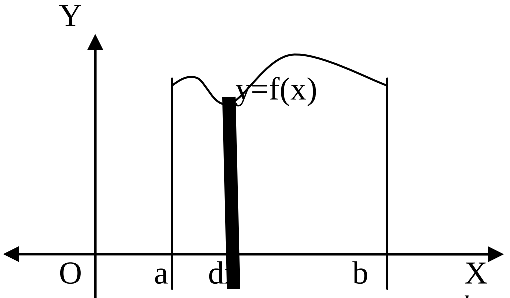
DEFINITE INTEGRAL

If $f(x)$ is a continuous function defined in the interval [a,b] and$F(x)$ is an anti-derivative of $f(x)$ $i.e.,\frac{dF(x)}{dx} = f(x)$,then the define integral of $f(x)$ over [a,b] is denoted by

$\int_a^b f(x)dx$ and is equal to $F(b) - F(a)$

$i.e., \int_a^b f(x)dx = F(b) - F(a)$

The constants a and b are called the limits of integration. ‘a’ is called the lower limit and ‘b’ the upper limit of integration.The interval [a,b] is called the interval of integration.



Geometrically, the definite integral $\int_a^b f(x)dx$ is the <u>AREA</u> of the region bounded by the curve $y = f(x)$ and the lines $x = a, x = b$ and x-axis.

EVALUATION OF DEFINITE INTEGRALS

To evaluate the definite integral $\int_a^b f(x)dx$ of a continuous function $f(x)$ defined on [a, b], we use the following steps.

Step-1:-Find the indefinite integral $\int f(x)dx$
Let $\int f(x)dx = F(x)$

Step-2:-Then, we have
$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$
i.e., definite integral is independent of the symbol of variable of integration.

3. $\int_a^b f(x)dx = \int_a^c f(x)dx = \int_c^b f(x)dx, a < c < b$

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx , a > 0$

5. $\int_{-a}^a f(x)dx = \left[\begin{array}{ll} 2\int_0^a f(x)dx, & iff(-x)=f(x) \\ 0, & iff(-x)=-f(x) \end{array}\right.$

6. $\int_0^{2a} f(x)dx = \left[\begin{array}{ll} 2\int_0^a f(x)dx, & iff(2a-x)=f(x) \\ 0, & iff(2a-x)=-f(x) \end{array}\right.$

Example:- Evaluate $\int_0^1 xtan^{-1}x\, dx$

Solution:-We have,$\int x tan^{-1}x\,dx = \frac{x^2}{2}tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$

$= \frac{x^2}{2}tan^{-1}x - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}dx$

$= \frac{x^2}{2}tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{x^2+1}$

$=\frac{x^2}{2}tan^{-1}x - \frac{x}{2} + \frac{1}{2}tan^{-1}x$

$=\frac{(x^2+1)}{2}tan^{-1}x - \frac{x}{2}$

$\therefore\ \int_0^1 x tan^{-1}x\,dx = \left[\frac{x^2+1}{2}tan^{-1}x - \frac{x}{2}\right]_0^1$

$=tan^{-1}1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$

Example:- Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$

Solution:-Let I=$\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$

$=\int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$=\int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx$

$\therefore$ 2I=I+I=$\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx = \int_0^{\pi/2} \frac{(sinx+\cos x)}{(sinx+cosx)}\,dx$

$=\int_0^{\pi/2} dx = x]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore$ I=$\frac{\pi}{4}$

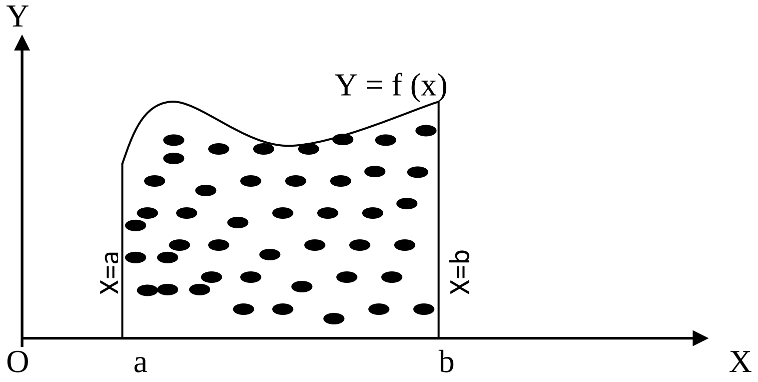
$\therefore\ \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx = \frac{\pi}{4}$

AREA UNDER PLANE CURVES

DEFINITION-1:-

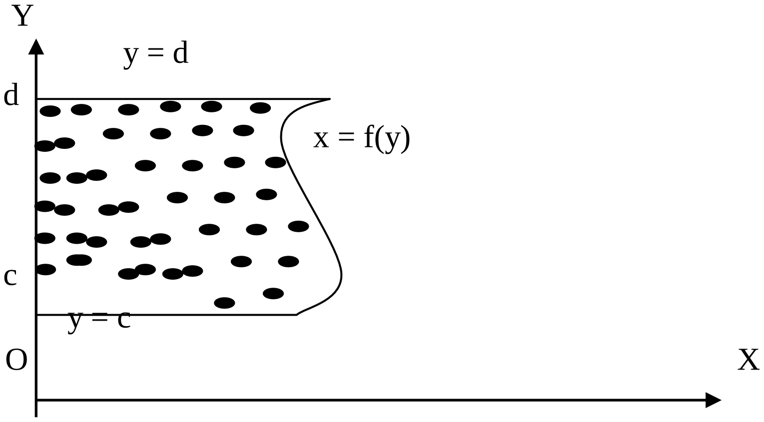
Area of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a, x = b$ is defined by

Area=$\left|\int_a^b y dx\right| = \left|\int_a^b f(x)dx\right|$



DEFINITION-2:-Area of the region bounded by the curve $x = f(y)$, the Y-axis and the lines $y = c, y = d$ is defined by

Area=$\left|\int_c^d xdy\right| = \left|\int_c^d f(y)dy\right|$



Example:-Find the area of the region bounded by the curve $y = e^{3x}, x$-axis and the lines $x = 4, x = 2$.

Solution:-The required area is defined by

$$A=\int_2^4 e^{3x}dx = \frac{1}{3}e^{3x}\Big]_2^4 = \frac{1}{3}(e^{12x} - e^{6x})$$

Example:-Find the area of the region bounded by the curve $xy = a^2, y$-axis and the lines $y = 2, y = 3$

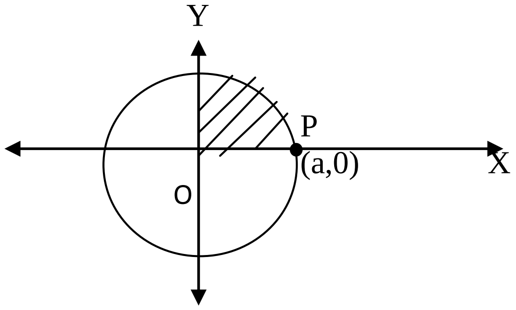
Solution:- $We\ have,\ xy = a^2 \Rightarrow x = \frac{a^2}{y}$

∴ The required area is defined by

$$A=\int_2^3 xdy = a^2\int_2^3 \frac{dy}{y} = [a^2\ln y]_2^3 = a^2(\ln 3 - \ln 2) = a^2\ln\left(\frac{3}{2}\right)$$

Example:-Find the area of the circle $x^2+y^2 = a^2$

Solution:-We observe that, $y = \sqrt{a^2-x^2}$ in the first quadrant.



∴ The area of the circle in the first quadrant is defined by,

$A_1 = \int_0^a \sqrt{a^2-x^2}\,dx$

As the circle is symmetrically situated about both X −axis and Y −axis, the area of the circle is defined by,

$$A=4\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=4\left|\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\frac{x}{a}\right|_0^a$$

$$=4\frac{a^2}{2}sin^{-1}1=2a^2\frac{\pi}{2}=\pi a^2.$$

DIFFERENTIAL EQUATIONS

DEFINITION:-An equation containing an independent variable (x), dependent variable (y) and differential co-efficients of dependent variable with respect to independent variable is called a differential equation.

For distance,

1. $\frac{dy}{dx}=\sin x+\cos x$
2. $\frac{dy}{dx}+2xy=x^3$
3. $y=x\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Are examples of differential equations.

ORDER OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest order derivative appearing in the equation.

Example:-In the equation, $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^x$,

The order of highest order derivative is 2. So, it is a differential equation of order 2.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the integral power of the highest order derivative occurring in the differential equation, after the equation has been expressed in a form free from radicals and fractions.

Example:-Consider the differential equation $\frac{d^3y}{dx^3}-6\left(\frac{dy}{dx}\right)^2-4y=0$

In this equation the power of highest order derivative is 1.So, it is a differential equation of degree 1.

Example:-Find the order and degree of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=K\frac{d^2y}{dx^2}$$

Solution:- By squaring both sides, the given differential equation can be written as $K^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$

The order of highest order derivative is 2.So, its order is 2.
Also, the power of the highest order derivative is 2.So, its degree is 2.

FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed by eliminating certain arbitrary constants from a relation in the independent variable, dependent variable and constants.

Example:-Form the differential equation of the family of curves $y = a\sin(bx + c)$, a and c being parameters.

Solution:-We have $y = a\sin(bx + c)$ -------------(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = ab\cos(bx + c) \qquad \text{----------------------(2)}$$

Differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -ab^2\sin(bx + c) \qquad \text{-------------------(3)}$$

Using (1) and (3), we get

$$\frac{d^2y}{dx^2} = -b^2y$$

$$\therefore \qquad \frac{d^2y}{dx^2} + b^2y = 0$$

This is the required differential equation.

Example:-Form the differential equation by eliminating the arbitrary constant in $y = Atan^{-1}x$.

Solution:-We have, $y = Atan^{-1}x$ ------------------(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{A}{1+x^2} \qquad \text{-------------------(2)}$$

Using (1) and (2), we get

$$\frac{dy}{dx} = \frac{y}{(1+x^2)tan^{-1}x}$$

$$\therefore\ (1 + x^2)tan^{-1}x\frac{dy}{dx} = y$$

This is the required differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is a relation $(like\ y = f(x)\ or\ f(x, y) = 0)$between the variables which satisfies the given differential equation.

GENERAL SOLUTION

The general solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.

PARTICULAR SOLUTION

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

SOLUTION OF FIRST ORDDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

We shall discuss some special methods to obtain the general solution of a first order and first degree differential equation.

1. Separation of variables

2. Linear Differential Equations

3. Exact Differential Equations

SEPARATION OF VARIABLES

If in a first order and first degree differential equation, it is possible to separate all functions of x and dx on one side, and all functions of y and dy on the other side of the equation, then the variables are said to be separable.Thus the general form of such an equation is $f(y)dy = \mathrm{g}(x)dx$
Then, Integrating both sides, we get

$\int f(y)dy = \int \mathrm{g}(x)dx + C$ as its solution.

Example:-Obtain the general solution of the differential equation

$$9y\frac{dy}{dx} + 4x = 0$$

Solution:- We have, $9y\frac{dy}{dx} + 4x = 0$

$$\Rightarrow 9y\frac{dy}{dx} = -4x$$

$$\Rightarrow 9y\,dy = -4x\,dx$$

Integrating both sides, we get

$$9\int ydy = -4\int xdx$$

$$\Rightarrow \frac{9}{2}\cdot y^2 = \frac{-4}{2}x^2 + K$$

$\Rightarrow 9y^2 = -4x^2 + C$ (Where C=2K)

$$\Rightarrow 4x^2 + 9y^2 = C$$

This is the required solution

LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear, if the dependent variable and its differential co-efficients occurring in the equation are of first degree only and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q, \quad \text{-----------------}(1)$$

Where P and Q are functions of x.

To solve linear differential equation of the form (1),
at first find the Integrating factor = $e^{\int P\,dx}$ -------------------(2)

It is important to remember that

I.F = $e^{\int P \cdot dx}$

Then, the general solution of the differential equation (1) is

$$y.(I.F) = \int Q.(I.F)dx + C \quad \text{-------------------}(3)$$

Example:-Solve $\frac{dy}{dx} + y \sec x = \tan x$

Solution:-The given differential equation is

$$\frac{dy}{dx} + (\sec x)y = \tan x \quad \text{-----------------}(1)$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \sec x$ and $Q = \tan x$

∴ I.F= $e^{\int P \cdot dx} = e^{\int \sec x dx}$ =$e^{\ln(\sec x + \tan x)}$

So, I.F=$\sec x + \tan x$

∴ The general solution of the equation (1) is

$$y.(I.F) = \int Q(I.F)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \int (\tan x\ \sec x + tan^2 x)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \int (\tan x\ \sec x + sec^2 x - 1)dx + C$$
$$\Rightarrow \quad y\,(\sec x + \tan x) = \sec x + \tan x - x + C$$

This is the required solution.

Example:-Solve:$(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

Solution:-The given differential equation can be written as

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2$$

$$\Rightarrow \frac{dy}{dx}+\frac{2x}{1+x^2}\cdot y=\frac{4x^2}{1+x^2} \quad \text{----------------(1)}$$

This is a linear equation of the form $\frac{dy}{dx}+Py=Q$,

Where $P=\frac{2x}{1+x^2}$ and $Q=\frac{4x^2}{1+x^2}$

We have, I.F = $e^{\int P\cdot dx}$ = $e^{\int 2x/(1+x^2)dx}$ = $e^{\ln(1+x^2)}=1+x^2$ ---------------(2)

∴The general solution of the given differential equation (1) is

$$y.(I.F)=\int Q.(I.F)dx+C$$

$$\Rightarrow \quad y(1+x^2)=\int \frac{4x^2}{1+x^2}.(1+x^2)dx+C$$

$$\Rightarrow \quad y(1+x^2)=4\int x^2dx+C$$

$$\Rightarrow \quad y(1+x^2)=\frac{4}{3}x^3+C$$

This is the required solution

EXACT DIFFERENTIAL EQUATIONS

DEFINITION:- A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

METHOD OF SOLUTION:-

The general solution of an exact differentia equation $Mdx+Ndy=0$ is

$$\underset{(y=\text{constant})}{\int Mdx}+\int (terms\ of\ N\ not\ containing\ x)dy=C,$$

Provided $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Example:- Solve;$(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$.

Solution:-The given differential equation is of the form $Mdx+Ndy=0$.

Where, $M=x^2-4xy-2y^2$ and $N=y^2-4xy-2x^2$

We have $\frac{\partial M}{\partial y}=-4x-4y=\frac{\partial N}{\partial x}$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, so the given differential equation is exact.

∴ The general solution of the given exact differential equation is

$$\underset{(y=\text{constant})}{\int Mdx}+\int (terms\ of\ N\ free\ from\ x)dy=C$$

$$\Rightarrow \quad \underset{(y=\text{constant})}{\int (x^2-4xy-2y^2)dx}+\int y^2dy=C$$

$\Rightarrow \quad \dfrac{x^3}{3} - 2x^2y - 2xy^2 + \dfrac{y^3}{3} = C$

$\Rightarrow \quad x^3 - 6x^2y - 6xy^2 + y^3 = C.$

This is the required solution.

Example:- Solve;$(x^2 - ay)dx = (ax - y^2)dy$

Solution:-The given differential equation can be written as

$(x^2 - ay)dx + (y^2 - ax)dy = 0$ ----------------(1)

Which is of the form $Mdx + Ndy = 0$,

Where,$M = x^2 - ay$ and $N = y^2 - ax.$

We have $\frac{\partial M}{\partial y} = -a \; and \; \frac{\partial N}{\partial x} = -a$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation (1) is exact.

∴ The solution of (1) is $\int (x^2 - ay)dx + \int y^2 dy = C$

(y=constant)

$\Rightarrow \dfrac{x^3}{3} - axy + \dfrac{y^3}{3} = C$

$\Rightarrow x^3 - 3axy + y^3 = C,$

Which is the required solution.

Example:- Solve; $ye^{xy}dx + (xe^{xy} + 2y)dy = 0.$

Solution:- The given differential equation is $ye^{xy}dx + (xe^{xy} + 2y)dy = 0,$

Which is of the form $Mdx + Ndy = 0.$

Where,$M = ye^{xy}$ and $N = xe^{xy} + 2y$

We have $\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$

So the given equation is exact and its solution is

$\int ye^{xy}dx + \int 2y dy = C.$

(y=constant)

$\Rightarrow e^{xy} + y^2 = C$

Example:- Solve; $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$

Solution:- The given equation is of the form $Mdx + Ndy = 0,$

Where,$M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$

We have $\frac{\partial M}{\partial y} = 12xy = \frac{\partial N}{\partial x}$.

So the given equation is exact and its solution is

$\int (3x^2 + 6xy^2)dx + \int (4y^3)dy = C$

(y=constant)

$\Rightarrow \dfrac{3x^3}{3} + \dfrac{6}{2}x^2y^2 + \dfrac{4}{4}y^4 = C$

$\Rightarrow x^3 + 3x^2y^2 + y^4 = C$

This is the required solution.

VECTORS

INTRODUCTION:-

In our real life situation we deal with physical quantities such as distance, speed, temperature, volume etc. These quantities are sufficient to describe change of position, rate of change of position, body temperature or temperature of a certain place and space occupied in a confined portion respectively.

We have also come across physical quantities such as displacement, velocity, acceleration, momentum etc, which are of different type in comparison to above.

Consider the figure-1, where A, B, C are at a distance 4k.m. from P. If we start from P, then covering 4k.m. distance is not sufficient to describe the destination where we reach after the travel, So here the end point plays an important role giving rise the need of direction. So we need to study about direction of a quantity, along with magnitude.

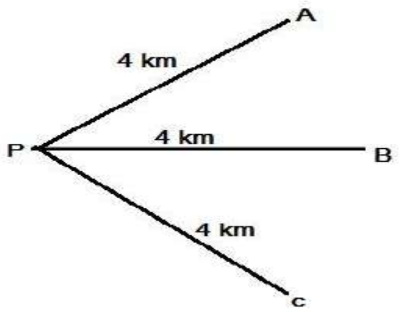


Fig - 1

OBJECTIVE

After completion of the topic you are able to :-

- i) Define and distinguish between scalars and vectors.
- ii) Represent a vector as directed line segment.
- iii) Classify vectors in to different types.
- iv) Resolve vector along two or three mutually perpendicular axes.
- v) Define dot product of two vectors and explain its geometrical meaning.
- vi) Define cross product of two vectors and apply it to find area of triangle and parallelogram.

Expected background knowledge

- i) Knowledge of plane and co-ordinate geometry
- ii) Trigonometry.

Scalars and vectors

All the physical quantities can be divided into two types.

- i) Scalar quantity or Scalar.
- ii) Vector quantity or Vector.

Scalar quantity: - The physical quantities which requires only magnitude for its complete specification is called as scalar quantities.

Examples: - Speed, mass, distance, velocity, volume etc.

Vector: - A directed line segment is called as vector.

Vector quantities:- A physical quantity which requires both magnitude & direction for its complete specification and satisfies the law of vector addition is called as vector quantities.
Examples: - Displacement, force, acceleration, velocity, momentum etc.

Representation of vector:- A vector is a directed line segment $\overrightarrow{AB}$where A is the initial point and B is the terminal point and direction is from A to B. (see fig-2).

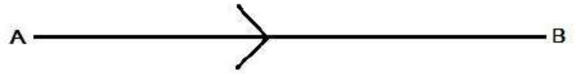


Fig - 2

Similarly $\overrightarrow{BA}$ is a directed line which represents a vector having initial point B and terminal point A.

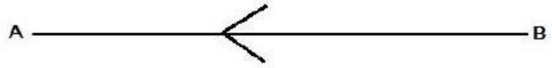


Fig - 3

Notation: - A vector quantity is always represented by an arrow (→) mark over it or by bar (−) over it. For example $\overrightarrow{AB}$. It is also represented by a single small letter with an arrow or bar mark over it. For example $\vec{a}$.

Magnitude of a vector: - Magnitude or modulus of a vector is the length of the vector. It is a scalar quantity.

Magnitude of $\overrightarrow{AB}$ = $|\overrightarrow{AB}|$= Length AB. = AB

Types of Vector: - Vectors are of following types.

1) **Null vector or zero vector or void vector: -** A vector having zero magnitude and arbitrary direction is called as a null vector and is denoted by $\vec{0}$.

Clearly, a null vector has no definite direction. If $\vec{a} = \overrightarrow{AB}$, then $\vec{a}$ is a null (or zero) vector iff $|\vec{a}|$= 0 i.e. if $|\overrightarrow{AB}|$= 0

For a null vector initial and terminal points are same.

2) Proper vector: - Any non zero vector is called as a proper vector. If $|\vec{a}|\neq 0$ then $\vec{a}$ is a proper vector.

3) Unit vector : - A vector whose magnitude is unity is called a unit vector. Unit vectors are denoted by a small letter with ^ over it. For example $\hat{a}$. $|\hat{a}|$= 1

Note: - The unit vector along the direction of a vector $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

4) Co-initial vectors:- Vectors having the same initial point are called co-initial vector.

In figure-4, $\overrightarrow{OA}$,$\overrightarrow{OB}$,$\overrightarrow{OC}$,$\overrightarrow{OD}$ and $\overrightarrow{OE}$ are Co-initial vectors.

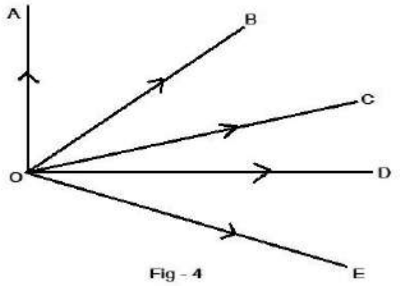


Fig - 4

5) Like and unlike vectors: - Vectors are said to be like if they have same direction and unlike if they have opposite direction.

6) Co-Linear vectors:- Vectors are said to be co-linear or parallal if they have the same line of action. In f figure-5 $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear.

Fig - 5

7) Parallel vectors: - Vectors are said to be parallel if they have same line of action or have line of action parallel to one another. In fig-6 the vectors are parallel to each other.

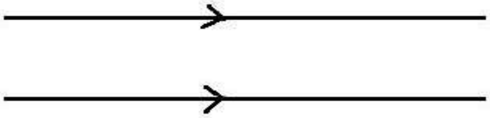


Fig - 6

8) Co-planner Vectors: - Vectors are said to be co-planner if they lies on the same plane. In fig-7 vector $\vec{a},\vec{b}\ and\ \vec{c}$ are coplanner.

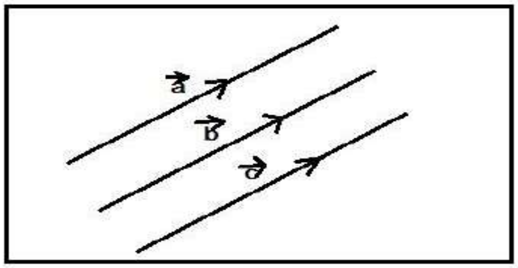


Fig - 7

9) Negative of a vector: - A vector having same magnitude but opposite in direction to that of a given vector is called negative of that vector. If $\overrightarrow{a}$ is any vector then negative vector of it is written as -$\overrightarrow{a}$ and $|\vec{a}|$= $|-\vec{a}|$ but both have direction opposite to each other as shown in fig-8.

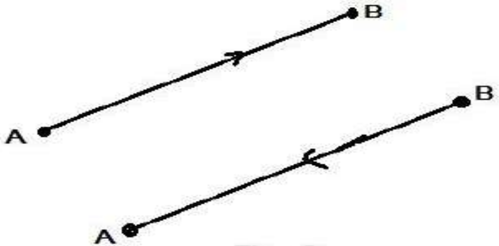


Fig - 8

10) Equal Vectors: - Two vectors are said to be equal if they have same magnitude as well as same direction.

Thus $\vec{a}$ = $\vec{b}$

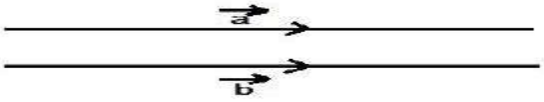


Fig - 9

Remarks:- Two vectors can not be equal

i) If they have different magnitude .

ii) If they have inclined supports.

iii) If they have different sense.

Vector operations

Addition of vectors: -

Triangle law of vector addition: - The law states that If two vectors are represented by the two sides of a triangle taken in same order their sum or resultant is represented by the 3rd side of the triangle with direction in reverse order.

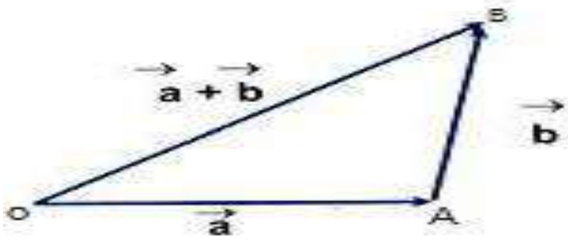


Fig – 10

As shown in figure-10 $\vec{a}$ and $\vec{b}$ are two vectors represented by two sides OA and AB of a triangle ABC in same order. Then the sum $\vec{a} + \vec{b}$ is represented by the third side OB taken in reverse order i.e. the vector $\vec{a}$ is represented by the directed segment $\overrightarrow{OA}$ and the vector $\vec{b}$ be the directed segment $\overrightarrow{AB}$, so that the terminal point A of $\vec{a}$ is the initial point of $\vec{b}$. Then $\overrightarrow{OB}$ represents the sum (or resultant) $(\vec{a} + \overrightarrow{b)}$. Thus $\overrightarrow{OB} = \vec{a} + \vec{b}$

Note-1 – The method of drawing a triangle in order to define the vector sum $(\vec{a} + \overrightarrow{b)}$ is called triangle law of addition of the vectors.

Note-2 – Since any side of a triangle is less than the sum of the other two sides

$|\overrightarrow{OB}| \neq |\overrightarrow{OA}| + |\overrightarrow{AB}|$

Parallelogram law of vector addition: - If $\vec{a}$ and $\vec{b}$ are two vectors represented by two adjacent side of a parallelogram in magnitude and direction, then their sum (resultant) is represented in magnitude and direction by the diagonal which is passing through the common initial point of the two vectors. As shown in fig-II if OA is $\vec{a}$ and AB is $\vec{b}$ then OB diagonal represent $\vec{a} + \vec{b}$.

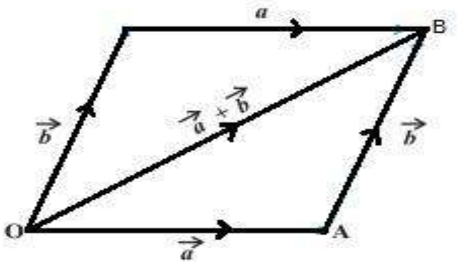


Fig - 11

i.e. $\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{AB}$

Polygon law of vector addition: - If $\vec{a}$, $\vec{b}, \vec{c}$ and $\vec{d}$ are the four sides of a polygon in same order then their sum is represented by the last side of the polygon taken in opposite order as shown in figure-12.

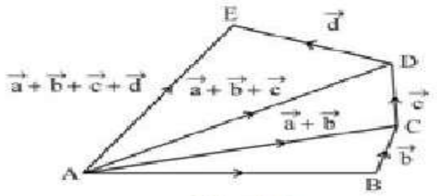


Fig - 12

Subtraction of two vectors

If $\vec{a}$ and $\vec{b}$ are two given vectors then the subtraction of $\vec{b}$ from $\vec{a}$ denoted by $\vec{a}$ - $\vec{b}$ is defined as addition of -$\vec{b}$ with $\vec{a}$. i.e. $\vec{a}$ - $\vec{b}$ =$\vec{a}$+ (-$\vec{b}$).

Properties of vector addition:- i) Vector addition is commutative i.e. if $\vec{a}$ & $\vec{b}$ are any two vectors then:-

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

ii) Vector addition is associative i.e. if $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors, then $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

iii) Existence of additive identity i.e. for any vector $\vec{a}$, $\vec{0}$ is the additive identity i.e. $\vec{0} + \vec{a} = \vec{a} + \vec{0} = \vec{a}$ where $\vec{0}$ is a null vector.

iv) Existence of additive Inverse :- If $\vec{a}$ is any non zero vector then -$\vec{a}$ is the additive inverse of $\vec{a}$, so that $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

Multiplication of a vector by a scalar : -

If $\vec{a}$ is a vector and k is a nonzero scalar then the multiplication of the vector $\vec{a}$ by the scalar k is a vector denoted by k$\vec{a}$ or $\vec{a}$k whose magnitude $|k|$ times that of $\vec{a}$.

i.e k$\vec{a}$ = $|k|$ x $|\vec{a}|$

= k x $|\vec{a}|$ if $k \geq 0$.

= (-k) x $|\vec{a}|$ if k<0 .

The direction of k$\vec{a}$ is same as that of $\vec{a}$ if k is positive and opposite as that of $\vec{a}$ if k is negative.

k$\vec{a}$ and $\vec{a}$ are always parallel to each other.

Properties of scalar multiplication of vectors :-

If h and k are scalars and $\vec{a}$ and $\vec{b}$ are given vectors then

i) k $(\vec{a} + \vec{b})$ = k$\vec{a}$ + k$\vec{b}$

ii) (h+k)$\vec{a}$ = h$\vec{a}$ + k$\vec{a}$, (Distributive law)

iii) (hk)$\vec{a}$ = h(k$\vec{a}$), (Associative law)

iv) 1.$\vec{a}$= $\vec{a}$

v) 0. $\vec{a}$ = $\vec{0}$

Position Vector of a point

Let O be a fixed point called origin, let P be any other point, then the vector $\overrightarrow{OP}$ is called position vector of the point P relative to O and is denoted by $\vec{p}$.

As shown in figure-13, let AB be any vector , then applying triangle law of addition we have

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ where $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

=> $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

= (Position vector of B)–(Position vector of A)

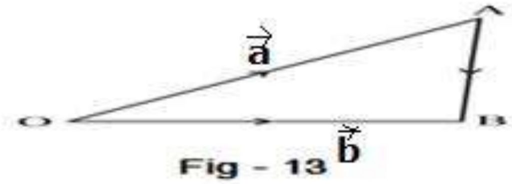


Fig - 13

Section Formula:- Let A and B be two points with position vector $\vec{a}$ and $\vec{b}$ respectively and P be a point on line segment AB , dividing it in the ration m:n. internally. Then the position vector of P i.e.$\vec{r}$ is given by the formula: $\vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n}$

If P divides AB externally in the ratio m:n then $\vec{r} = \frac{m\vec{b}-n\vec{a}}{m-n}$

If P is the midpoint of AB then $\vec{r} = \frac{\vec{a}+\vec{b}}{2}$

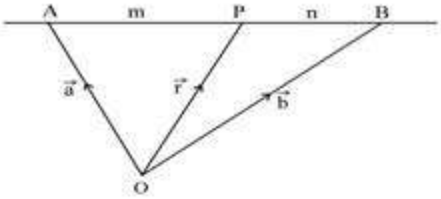


Fig - 14

Example-1 :- Prove that by vector method the medians of a triangle are concurrent.

Solution:- Let ABC be a triangle where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vector of A,B and C respectively. We have to show that the medians of this triangle are concurrent.

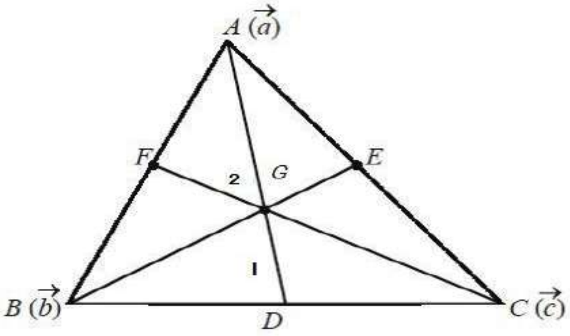


Fig - 15

Let AD , BE and CF are the three medians of the triangle.

Now as D be the midpoint of BC, so position vector of D i.e. $\vec{d} = \frac{\vec{b}+\vec{c}}{2}$.

Let G be any point of the median AD which divides AD in the ratio 2:1 . Then position vector of G is given by $\vec{g} = \frac{2\vec{d}+\vec{a}}{2+1} = \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right)+1\vec{a}}{3}$ (by applying section formula)

$$\Rightarrow \vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Let G' be a point which divides BE in the ratio 2:1 ,

Position vector of E is $\vec{e} = \frac{\vec{a}+\vec{c}}{2}$.

Then position vector of G' is given by $\overrightarrow{g'} = \frac{2\vec{e}+\vec{b}}{2+1} = \frac{\mathbf{2}\left(\frac{\vec{a}+\vec{c}}{2}\right)+1\vec{b}}{3}$ **(**by applying section formula)

$$\Rightarrow \overrightarrow{g'} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

As position vector of a point is unique , so G = G' .

Similarly if we take G'' be a point on CF dividing it in 2:1 ratio then the position vector of G'' will be same as that of G.

Hence G is the one point where three median meet.

$\therefore$ The three medians of a triangle are concurrent. (proved)

Example2: - Prove that i) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (It is known as Triangle Inequality).

ii) $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$

iii) $|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$

Proof:- Let O,A and B be three points, which are not collinear and then draw a triangle OAB.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, then by triangle law of addition we have $\overrightarrow{OB} = \vec{a} + \vec{b}$

From properties of triangle we know that the sum of any two sides of a triangle is greater than the third side.

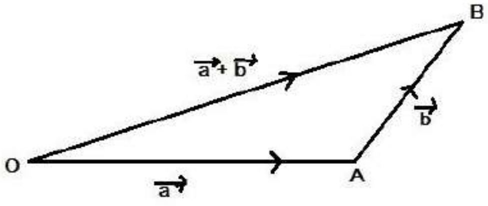


Fig - 16

- ⇨ OB < OA + AB
- ⇨ $|\overrightarrow{OB}| < |\overrightarrow{OA}| + |\overrightarrow{AB}|$
- ⇨ $|\vec{a}+\vec{b}| < |\vec{a}| + |\vec{b}|$ -------------------(1)

When O,A, B are collinear then

From figure-17 it is clear that

OB = OA + AB

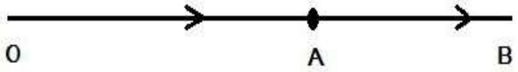


Fig-17

- ⇨ $|\overrightarrow{OB}| = |\overrightarrow{OA}| + |\overrightarrow{AB}|$
- ⇨ $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ ------------------(2)

From (1) and (2) we have,

$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (proved)

ii) $|\vec{a}| = |\vec{a}-\vec{b}+\vec{b}|$ --------------(1)

But $|(\vec{a}-\vec{b})+\vec{b}\ | \leq |\vec{a}-\vec{b}| + |\vec{b}|$ (From triangle inequality)--------(2)

From (1) and (2) we get $|\vec{a}| \leq |\vec{a}-\vec{b}| + |\vec{b}|$

⇨ $|\vec{a}| - |\vec{b}| \leq |\vec{a}-\vec{b}|$(proved)

iii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \leq |\vec{a}| + |-\vec{b}|$ (From triangle inequality)

$= |\vec{a}| + |\vec{b}|$ (as $|-\vec{b}| = |\vec{b}|$)

$|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (proved)

Components of vector in 2D

Let XOY be the co-ordinate plane and P(x,y) be any point in this plane.

The unit vector along direction of X axis i.e. $\overrightarrow{OX}$ is denoted by $\hat{\imath}$.

The unit vector along direction of Y axis i.e. $\overrightarrow{OY}$ is denoted by $\hat{\jmath}$.

Then from figure-18 it is clear that $\overrightarrow{OM} = x\hat{\imath}$ and $\overrightarrow{ON} = y\hat{\jmath}$.

So, the position vector of P is given by

$$\boxed{\overrightarrow{OP} = \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}}$$

And OP $=|\overrightarrow{OP}| = r = \sqrt{x^2+y^2}$

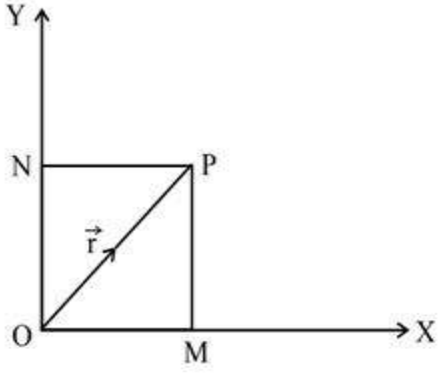


Fig-18

Representation of vector in component form in 2D

If $\overrightarrow{AB}$ is any vector having end points A(x_1, y_1) and B (x_2, y_2) , then it can be represented by

$\overrightarrow{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$

Components of vector in 3D

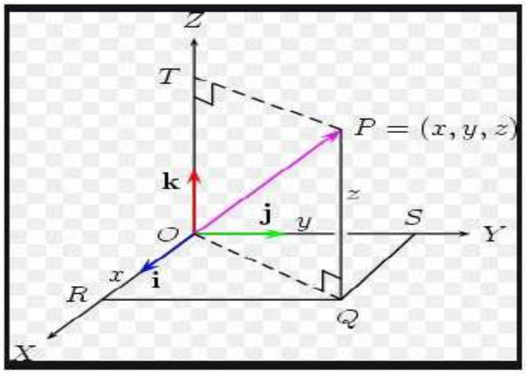


Fig-19

Let P(x,y,z) be a point in space and $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ be the unit vectors along X axis , Y axis and Z axis respectively. (as shown in fig-19)

Then the position vector of P is given by

$\overrightarrow{OP} = x\,\hat{\imath} + y\,\hat{\jmath} + \hat{k}$, The vectors $x\hat{\imath}$, $y\,\hat{\jmath}$, $z\hat{k}$ are called the components of $\overrightarrow{OP}$ along x–axis, y-axis and z-axis respectively.

And OP $= |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

Addition and scalar multiplication in terms of component form of vectors: -

For any vector $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

i) $\vec{a} + \vec{b} = (a_1 + b_1)\,\hat{\imath} + (a_2 + b_2)\hat{\jmath} + (a_3 + b_3)\hat{k}$

ii) $\vec{a} - \vec{b} = (a_1 - b_1)\,\hat{\imath} + (a_2 - b_2)\hat{\jmath} + (a_3 - b_3)\hat{k}$

iii) k $\vec{a} = ka_1\hat{\imath} + ka_2\hat{\jmath} + ka_3\hat{k}$, where K is a scalar.

iv) $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$\Leftrightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$

Representation of vector in component form in 3-D & Distance between two points:

If $\overrightarrow{AB}$ is any vector having end points A(x_1, y_1, z_1) and B (x_2, y_2, z_2) , then it can be represented by

$\overrightarrow{\boldsymbol{AB}}$ **= Position vector of B – Position vector of A**

$$= (x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})$$

$$= \boldsymbol{(x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}}$$

$$|\overrightarrow{\boldsymbol{AB}}| = \sqrt{(\boldsymbol{x_2} - \boldsymbol{x_1})^2 + (\boldsymbol{y_2} - \boldsymbol{y_1})^2 + (\boldsymbol{z_2} - \boldsymbol{z_1})^2}$$

Example 3:-

Show that the points A(2,6,3) , B(1,2,7) and c(3,10,-1) are collinear.

Solution:- From given data Position vector of A, $\overrightarrow{OA} = 2\,\hat{\imath} + 6\hat{\jmath} + 3\,\hat{k}$.

Position vector of B, $\overrightarrow{OB} = \hat{\imath} + 2\hat{\jmath} + 7\hat{k}$

Position vector of C, $\overrightarrow{OC} = 3\,\hat{\imath} + 10\hat{\jmath} - \hat{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1-2)\,\hat{\imath} + (2-6)\,\hat{\jmath} + (7-3)\,\hat{k} = -\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3-2)\,\hat{\imath} + (10 - 6)\,\hat{\jmath} + (-1-3)\,\hat{k} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$.

$= -(-\hat{\imath} - 4\hat{\jmath} + 4\hat{k}) = -\overrightarrow{AB}$

⇨ $\overrightarrow{AB} \parallel \overrightarrow{AC}$ or collinear.

∴ They have same support and common point A.
As ‘A’ is common to both vector , that proves A,B and C are collinear.

Example-4: - Prove that the points having position vector given by $2\,\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\,\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ form a right angled triangle. [2009(w)]

S0lution :- Let A,B and C be the vertices of a triangle with position vectors $2\,\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\,\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ respectively

Then . $\overrightarrow{AB}$ = Position vector of B – Position vector of A.

$= (1-2)\,\hat{\imath} + (-3 - (-1))\,\hat{\jmath} + (-5-1)\,\hat{k} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$.

$\overrightarrow{BC}$ = Position vector of C – Position vector of B.

$= (3-1)\,\hat{\imath} + (-4 - (-3))\,\hat{\jmath} + (-4-(-5))\,\hat{k} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$.

$\overrightarrow{AC}$ = Position vector of C – Position vector of A.

$= (3-2)\,\hat{\imath} + (-4 - (-1))\,\hat{\jmath} + (-4-1)\,\hat{k} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$.

Now AB = $|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

BC = $|\overrightarrow{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

AC = $|\overrightarrow{AC}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

From above $BC^2 + AC^2$ = 6+35 = 41 = AB^2.

Hence ABC is a right angled triangle.

Example-5 :- Find the unit vector in the direction of the vector $\vec{a} = 3\hat{\imath} - 4\hat{\jmath} + \hat{k}$. (2017-W)

Ans:- The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{\imath} - 4\hat{\jmath} + \hat{k}}{\sqrt{3^2 + (-4)^2 + 1^2}} = \frac{3\hat{\imath} - 4\hat{\jmath} + \hat{k}}{\sqrt{9+16+1}} = \frac{3}{\sqrt{26}}\hat{\imath} - \frac{4}{\sqrt{26}}\hat{\jmath} + \frac{1}{\sqrt{26}}\hat{k}.$$

Example-6 :- Find a unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = = \hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans:- Let $\vec{r} = \vec{a} + \overrightarrow{b} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + (\hat{\imath} - \hat{\jmath} + 3\hat{k}) = 2\,\hat{\imath} + 2\hat{k}$.

Unit vector along direction of $\vec{a} + \vec{b}$ is given by $= \frac{\vec{r}}{|\vec{r}|} = \frac{2\,\hat{\imath} + 2\hat{k}}{\sqrt{2^2+2^2}} = \frac{2\,\hat{\imath} + 2\hat{k}}{\sqrt{8}} = \frac{2}{\sqrt{8}}\hat{\imath} + \frac{2}{\sqrt{8}}\hat{k}$

$= \frac{2}{2\sqrt{2}}\hat{k}\hat{\imath} + \frac{2}{2\sqrt{2}}\widehat{\jmath} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$.

Angle between the vectors:-

As shown in figure-20 angle between two vectors $\overrightarrow{RS}$ and $\overrightarrow{PQ}$ can be determined as follows.

Let $\overrightarrow{OB}$ be a vector parallel to $\overrightarrow{RS}$ and $\overrightarrow{OA}$ is a vector parallel to $\overrightarrow{PQ}$ such that $\overrightarrow{OB}$ and $\overrightarrow{OA}$ intersect each other.

Then θ = ∟AOB = angle between $\overrightarrow{RS}$ and $\overrightarrow{PQ}$.

If θ = 0 then vectors are said to be parallel.

If θ = $\frac{\pi}{2}$ then vectors are said to be orthogonal or perpendicular.

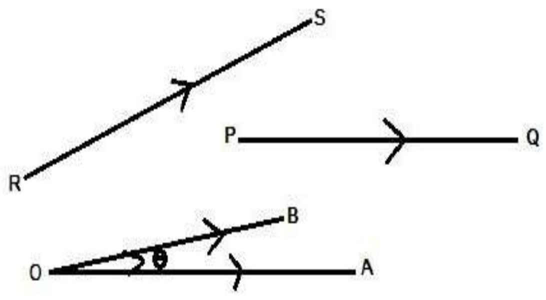


Fig-20

Dot Product or Scalar product of vectors

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are, a and b respectively denoted by $\vec{a}.\vec{b}$ is defined as the scalar abcosθ , where θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq \theta \leq \pi$.

$$\vec{a}.\vec{b} = |\vec{a}|\, |\vec{b}| \cos\theta = a\, b \cos\theta$$

Geometrical meaning of dot product

In figure21(a), $\vec{a}$ and $\vec{b}$ are two vectors having θ angle between them. Let M be the foot of the perpendicular drawn from B to OA.

Then OM is the Projection of $\vec{b}$ on $\vec{a}$ and from figure-21(a) it is clear that ,

|OM| = |OB| cosθ = |$\vec{b}$| cosθ.

Now $\vec{a}.\vec{b}$ = |$\vec{a}$| (|$\vec{b}$| cosθ) = |$\vec{a}$| x projection of $\vec{b}$ on $\vec{a}$

which gives projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Similarly we can write $\vec{a}.\vec{b}$ = |$\vec{a}$| |$\vec{b}$| cosθ

=|$\vec{b}$| (|$\vec{a}$| cosθ) = |$\vec{b}$| projection of $\vec{a}$ on $\vec{b}$.

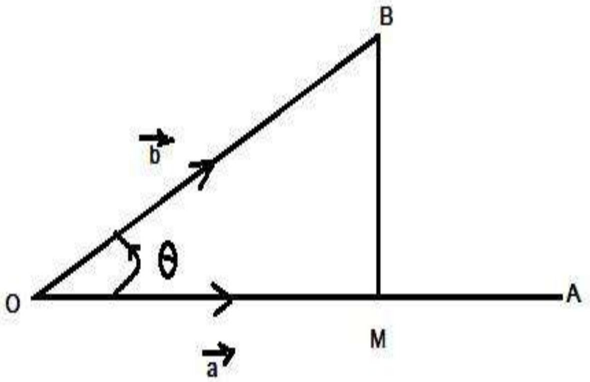


Fig-21

(a)

Similarly, let us draw a perpendicular from A on OB and let N be the foot of the perpendicular in fig-21(b).

Then ON = Projection of $\vec{a}$ on $\vec{b}$

and ON = OAcosθ =I$\vec{a}$I cosθ.

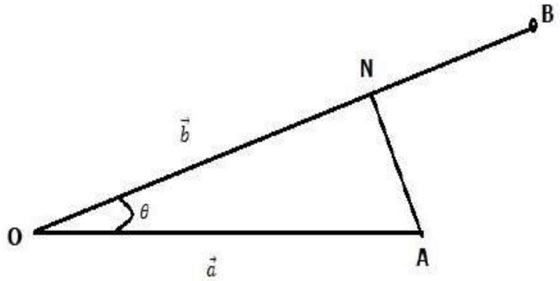


Fig-21(b)

<u>Properties of Dot product</u>

i) $\vec{a}.\vec{b}$ = $\vec{b}.\vec{a}$ (commutative)

ii) $\vec{a}.(\vec{b}+\vec{c})$ = $\vec{a}.\vec{b}$ + $\vec{a}.\vec{c}$ (Distributive)

iii) If $\vec{a}$ II $\vec{b}$, then $\vec{a}.\vec{b}$ = ab { as θ = 0 in this case cos 0 = 1 }

In particular $(\vec{a})^2$ = $\vec{a}\,.\vec{a}$ = I $\vec{a}$I^2

$$\boldsymbol{\hat{\imath}.\,\hat{\imath} = \hat{\jmath}.\,\hat{\jmath} = \hat{k}.\,\hat{k} = 1}$$

iv) If $\vec{a} \perp \vec{b}$, then $\vec{a}.\vec{b}$ = 0. { as θ = 90° in this case cos90º= 0 }

In particular $\boldsymbol{\hat{\imath}.\,\hat{\jmath} = \hat{\jmath}.\,\hat{k} = \hat{k}.\,\hat{\imath} = 0 = \hat{\jmath}.\hat{\imath} = \hat{k}.\hat{\jmath} = \hat{\imath}.\hat{k}}$

v) $\vec{a}.\vec{0}$ = $\vec{0}.\vec{a}$ = 0

vi) $(\vec{a}+\vec{b})$. $(\vec{a}-\vec{b})$ = I$\vec{a}$I^2- I$\vec{b}$I^2= a^2 - b^2 {Where I$\vec{a}$ I = a and I$\vec{b}$ I = b }

viii) Work done by a Force:- The work done by a force $\vec{F}$ acting on a body causing displacement $\vec{d}$ is given by W = $\vec{F}.\vec{d}$

<u>Dot product in terms of rectangular components</u>

For any vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ we have,

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ (by applying distributive(ii) , (iii) and (iv) successively)

<u>Angle between two non zero vectors</u>

For any two non zero vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, having θ is the angle between them we have,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \hat{a}.\hat{b} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}} \quad \textbf{(In terms of components.)}$$

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Condition of Perpendicularity: -

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are perpendicular to each other $\Leftrightarrow \boldsymbol{a_1 b_1 + a_2 b_2 + a_3 b_3 = 0}$

Condition of Parallelism :-

Two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are parallel to each other $\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$)

Scalar & vector projections of two vectors (Important formulae)

Scalar Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Vector Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\hat{a} = [\frac{\vec{b}.\vec{a}}{|\vec{a}|^2}]\vec{a}$

Scalar Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\hat{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b}$

Examples: -

Q.- 7. Find the value of p for which the vectors $3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$, $\hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are perpendicular to each other.

Solution:- Let $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$.

Here $a_1 = 3$, $a_2 = 2$, $a_3 = 9$

$b_1 = 1$, $b_2 = p$ & $b_3 = 3$

Given $\vec{a} \perp \vec{b} \Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

$\Rightarrow 3.1+2.p+9.3 = 0$

$\Rightarrow 3 + 2p + 27 = 0$

$\Rightarrow 2p = -30$

$\Rightarrow p = -15$ (Ans)

Q-8 Find the value of p for which the vectors $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$, $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are parallel to each other. (2014-W)

Solution:- Given $\vec{a} \parallel \vec{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \Leftrightarrow \frac{3}{1} = \frac{2}{p} = \frac{9}{3}$ { Taking 1st two terms}

$\Leftrightarrow 3 = \frac{2}{p} \Leftrightarrow p = \frac{2}{3}$ (Ans) {Note:- any two expression may be taken for finding p.}

Q-9 Find the scalar product of $3\hat{\imath} - 4\hat{\jmath}$ and $-2\hat{\imath} + \hat{\jmath}$. (2015-S)

Solution:- $(3\hat{\imath} - 4\hat{\jmath}) . (-2\hat{\imath} + \hat{\jmath}) = (3\times(-2)) + ((-4)\times1) = (-6)+(-4) = -10$

Q-10 Find the angle between the vectors $5\hat{\imath}\ +3\hat{\jmath} + 4\hat{k}$ and $6\hat{\imath}-8\hat{\jmath}-\hat{k}$. (2015-W)

Solution:- Let $\vec{a}= 5\hat{\imath}\ +3\hat{\jmath} + 4\hat{k}$ and $\vec{b}= 6\hat{\imath}-8\hat{\jmath}-\hat{k}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then $\theta\ = cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{{a_1}^2+{a_2}^2+{a_3}^2}\sqrt{{b_1}^2+{b_2}^2+{b_3}^2}}\right)$

$$= cos^{-1}\left(\frac{5.6+3.(-8)+4.(-1)}{\sqrt{5^2+3^2+4^2}\sqrt{6^2+(-8)^2+(-1)^2}}\right) = cos^{-1}\left(\frac{30-24-4}{\sqrt{50}\sqrt{101}}\right) = cos^{-1}\left(\frac{2}{\sqrt{50}\sqrt{101}}\right)$$

Q-11 Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where,

$\vec{a}= \hat{\imath}\ -\hat{\jmath}-\hat{k}$ and $\vec{b}= 3\hat{\imath}+ \hat{\jmath}+3\hat{k}$. { 2013-W ,2017-W, 2017-S}

Solution:- Scalar Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{\mathrm{I}\vec{b}\mathrm{I}} = \frac{1.3+(-1).1+(-1).3}{\sqrt{3^2+1^2+3^2}} = \frac{3-1-3}{\sqrt{19}} = \frac{-1}{\sqrt{19}}$

Vector Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{\boldsymbol{a}}.\vec{\boldsymbol{b}}}{\mathbf{I}\vec{\boldsymbol{b}}\mathbf{I}^2}\vec{b} = \frac{1.3+(-1).1+(-1).3}{(\sqrt{3^2+1^2+3^2})^2}\ (3\hat{\imath}+ \hat{\jmath}+3\hat{k}\ )$

$= \frac{3-1-3}{19}\ (3\hat{\imath}+ \hat{\jmath}+3\hat{k}\ ) \quad = \frac{-1}{19}\ (3\hat{\imath}+ \hat{\jmath}+3\hat{k}\ )$

Q-12 Find the scalar and vector projection of $\vec{b}$ on $\vec{a}$ where,

$\vec{a}=3\ \hat{\imath}\ +\hat{\jmath}-2\hat{k}$ and $\vec{b}= 2\hat{\imath}+3\hat{\jmath}- 4\hat{k}$. { 2015-S}

Solution: - Scalar Projection of $\vec{b}$ on $\vec{a}$ $= \frac{\vec{a}.\vec{b}}{\mathrm{I}\vec{a}\mathrm{I}} = \frac{3.2+1.3+(-2).(-4)}{\sqrt{3^2+1^2+(-2)^2}} = \frac{6+3+8}{\sqrt{14}} = \frac{17}{\sqrt{14}}$

Vector Projection of $\vec{b}$ on $\vec{a}$ $= \frac{\vec{\boldsymbol{a}}.\vec{\boldsymbol{b}}}{\mathbf{I}\vec{\boldsymbol{a}}\mathbf{I}^2}\vec{a} = \frac{3.2+1.3+(-2).(-4)}{(\sqrt{3^2+1^2+(-2)^2})^2}\ (3\ \hat{\imath}\ +\hat{\jmath}-2\hat{k})$

$= \frac{17}{14}\ (3\hat{\imath}+ \hat{\jmath}- 2\hat{k}\ ).$

Q-13 If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, then prove that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\ \vec{b} - \vec{c}\ )$

Proof:- Given $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

⇨ $(\vec{a}.\vec{b}\ ) – (\vec{a}.\vec{c}\ ) = \vec{0}$ $=> \vec{a}\ .\ (\vec{b} - \vec{c}\ ) = \vec{0}$ { applying distributive property}

Dot product of above two vector is zero indicates the following conditions

$\vec{a} = \vec{0}$ or $\vec{b} - \vec{c}\ = \vec{o}$ or $\vec{a} \perp (\ \vec{b} - \vec{c}\ )$

⇨ $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\ \vec{b} - \vec{c}\ )$ (proved)

Example:-14 Find the work done by the force $\vec{F}= \hat{\imath}+\hat{\jmath}- \hat{k}$. acting on a particle if the particle is displace A from A(3,3,3) to B(4,4,4).

Ans:- Let O be the origin, then

Position vector of A $\overrightarrow{OA} = 3\hat{\imath}+3\hat{\jmath}+3\hat{k}$

Position vector of B $\overrightarrow{OB} = 4\hat{\imath}+4\hat{\jmath}+ 4\hat{k}$

Then displacement is given by, $\vec{d} = \overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) = (4\hat{\imath}+4\hat{\jmath}+4\hat{k}) - (3\hat{\imath}+3\hat{\jmath}+3\hat{k}) = \hat{\imath}+\hat{\jmath}+\hat{k}$.

So work done by the force W = $\vec{F}.\vec{d} = \vec{F}.\overrightarrow{AB} = (\hat{\imath}+\hat{\jmath}-\hat{k}).(\hat{\imath}+\hat{\jmath}+\hat{k})$

$= 1.1+1.1+(-1).1 = 1$ units

Example:-15 If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is the angle between them then prove that

$$\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$$

Proof: - $(|\hat{a}-\hat{b}|)^2 = (\hat{a}-\hat{b}).(\hat{a}-\hat{b}) = (\hat{a}.\hat{a}) - (\hat{a}.\hat{b}) - (\hat{b}.\hat{a}) + (\hat{b}.\hat{b})$ { Distributive property}

$= (|\hat{a}|)^2 - (\hat{a}.\hat{b}) - (\hat{a}.\hat{b}) + (|\hat{b}|)^2$ {commutative property}

$= 1^2 - 2\,\hat{a}.\hat{b} + 1^2$ { as $\hat{a}$ and $\hat{b}$ are unit vectors so their magnitudes are 1}

$= 2 - 2\,\hat{a}.\hat{b} = 2(1 - \hat{a}.\hat{b})$

$= 2(1 - |\hat{a}|.|\hat{b}|\cos\theta)$ { as θ is the angle between $\hat{a}$ and $\hat{b}$ }

$= 2(1 - 1.1.\cos\theta)$

$= 2(1 - \cos\theta) = 2.2\, sin^2\frac{\theta}{2}$

Taking square root of both sides we have $|\hat{a}-\hat{b}| = 2\sin\frac{\theta}{2}$

⇨ $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$ (proved)

Example:-16 If the sum of two unit vectors is a unit vector. Then show that the magnitude of their difference is $\sqrt{3}$.

Proof:-$\hat{a}$,$\hat{b}$ and $\hat{c}$ are three unit vectors such that $\hat{a}+\hat{b} = \hat{c}$

Squaring both sides we have,

=> $(|\hat{a}+\hat{b}|)^2 = (|\hat{c}|)^2$

=> $(|\hat{a}|)^2 + (|\hat{b}|)^2 + 2\,\hat{a}.\hat{b} = 1^2$

=> $1^2 + 1^2 + 2|\hat{a}||\hat{b}|\cos\theta = 1$ { where θ is the angle between $\hat{a}$ and $\hat{b}$ }

=> $1 + 1 + 2\cos\theta = 1$

=> $2\cos\theta = -1$

=> $\cos\theta = \frac{-1}{2}$

Now we have to find the magnitude of their difference i.e $\hat{a}-\hat{b}$.

So $(|\hat{a}-\hat{b}|)^2 = (|\hat{a}|)^2 + (|\hat{b}|)^2 - 2\,\hat{a}.\hat{b} = 1^2 + 1^2 - 2|\hat{a}||\hat{b}|\cos\theta$

$= 2 - 2\cos\theta = 2 - 2(\frac{-1}{2}) = 2 - (-1) = 3$

$\therefore |\hat{a}-\hat{b}| = \sqrt{3}$ (Proved)

Vector Product or Cross Product

If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them , then the vector product of these two vectors denoted by $\vec{a} \times \vec{b}$ is defined as

$$\boxed{\vec{a} \times \vec{b} = |\vec{a}|.|\vec{b}| \sin\theta \hat{n}}$$

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

As shown in figure-21 the direction of $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$ and $\vec{b}$.

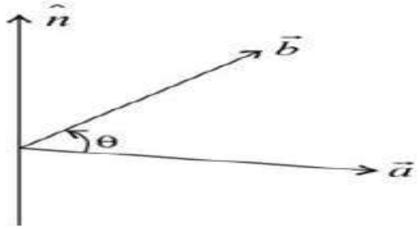


Fig-22

Properties of cross product

i) Vector product is not commutative $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

ii) For any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

iii) For any scalar m, $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

iii) Distributive $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

iv) Vector product of two parallel or collinear vectors is zero.

$\vec{a} \times \vec{a} = \vec{0}$ and if $\vec{a} \parallel \vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$ { as $\theta = 0$ or 180^{o} => $\sin\theta = 0$ }

Using this property we have,

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

v) Vector product of orthonormal unit vectors form a right handed system.

As shown in figure- 23 the three mutually perpendicular unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ form a right handed system , i.e. $\hat{\imath} \times \hat{\jmath} = \hat{k} = -(\hat{\jmath} \times \hat{\imath})$

(as θ =90 , then $\sin\theta = 1$)

$\hat{\jmath} \times \hat{k} = \hat{\imath} = -(\hat{k} \times \hat{\jmath})$

$\hat{k} \times \hat{\imath} = \hat{\jmath} = -(\hat{\imath} \times \hat{k})$

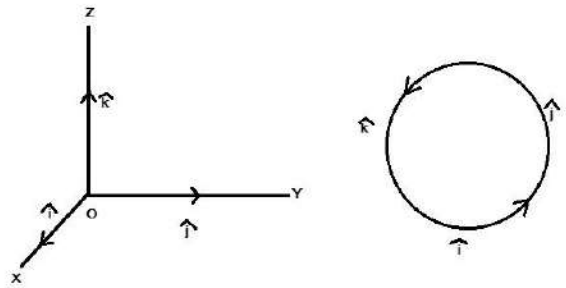


Fig-23

Unit vector perpendicular to two vectors:- Unit vector perpendicular to two given vectors $\vec{a}$and $\vec{b}$ is given by $\boldsymbol{\hat{n}} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

Angle between two vectors

Let θ be the angle between $\vec{a}$and $\vec{b}$. Then $\vec{a} \times \vec{b} = (|\vec{a}|.|\vec{b}|\sin\theta)\,\hat{n}$.

Taking modulus of both sides we have,

$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|\sin\theta$

⇨ $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|.|\vec{b}|}$

Hence $\boldsymbol{\theta} = \boldsymbol{sin^{-1}}\left\{\frac{|\vec{a}\times\vec{b}|}{|\vec{a}|.|\vec{b}|}\right\}$

Geometrical Interpretation of vector product or cross product

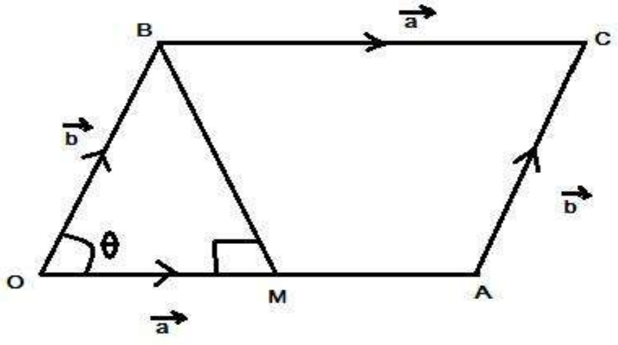


Fig-24

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Then $\vec{a} \times \vec{b} = (|\vec{a}|.|\vec{b}|\sin\theta)\,\hat{n}$

$= (|\vec{a}|).(|\vec{b}|\sin\theta)\hat{n}$

From fig-24 below it is clear that

BM = OB sin θ = $|\vec{b}|\sin\theta = |\vec{a}|\,|BM|\,\hat{n}$

{ as $\sin\theta$ = BM/OB &$\overrightarrow{OB} = \vec{b}$}

Now $|\vec{a} \times \vec{b}| = |\vec{a}|\,|BM|\,|\hat{n}|$ = OA.

BM = Area of the parallelogram with side $\vec{a}$ and $\vec{b}$.

Therefore the magnitude of cross product of two vectors is equal to area of the parallelogram formed by these vectors as two adjacent sides.

From this it can be concluded that area of △ ABC = ½ $|\overrightarrow{AB} . \overrightarrow{AC}|$

Application of cross product

1. Moment of a force about a point ($\vec{M}$) :-Let O be any point and Let $\vec{r}$ be the position vector w.r.t. O of any point 'P' on the line of action of the force $\vec{F}$, then the moment or torque of the force F about origin 'O' is given by

$\vec{M} = \vec{r} \times \vec{F}$

2. If $\vec{a}$and $\vec{b}$ represent two adjacent sides of a triangle then the area of the triangle is given by $\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

3. If $\vec{a}$and $\vec{b}$ represent two adjacent sides of a parallelogram then area of the parallelogram is given by $\Delta = |\vec{a} \times \vec{b}|$ Sq. unit

4. If $\vec{a}$and $\vec{b}$ represent two diagonals of a parallelogram then area of the parallelogram is given by $\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$ Sq. unit

Vector product in component form : -

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$.

$$\vec{a} \times \vec{b} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$= a_1b_1(\hat{\imath}\times\hat{\imath}) + a_1b_2(\hat{\imath}\times\hat{\jmath}) + a_1b_3(\hat{\imath}\times\hat{k}) + a_2b_1(\hat{\jmath}\times\hat{\imath}) + a_2b_2(\hat{\jmath}\times\hat{\jmath}) + a_2b_3(\hat{\jmath}\times\hat{k})$$

$$+ a_3b_1(\hat{k}\times\hat{\imath}) + a_3b_2(\hat{k}\times\hat{\jmath}) + a_3b_3(\hat{k}\times\hat{k})$$

{ using properties $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = \vec{0}$, $\hat{\imath}\times\hat{\jmath} = \hat{k} = -(\hat{\jmath}\times\hat{\imath})$, $\hat{\jmath}\times\hat{k} = \hat{\imath} = -(\hat{k}\times\hat{\jmath})$ and $\hat{k}\times\hat{\imath} = \hat{\jmath} = -(\hat{\imath}\times\hat{k})$ }

$$= (a_2b_3 - a_3b_2)\,\hat{\imath} + (a_3b_1 - a_1b_3)\,\hat{\jmath} + (a_1b_2 - a_2b_1)\,\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{i.e. } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Condition of Co-planarity

If three vectors $\vec{a}, \vec{b}$and $\vec{c}$ lies on the same plane then the perpendicular to $\vec{a}$ *and* $\vec{b}$ must be perpendicular to $\vec{c}$.

In particular $(\vec{a} \times \vec{b}) \perp \vec{c}$ => $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

In component form if $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$,$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

Then $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

- ⇨ $(a_2b_3 - a_3b_2)\,c_1 + (a_3b_1 - a_1b_3)\,c_2 + (a_1b_2 - a_2b_1)\,c_3 = 0$
- ⇨ $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$ (interchanging rows two times R_1 *and* R_2, *then* R_2 *and* R_3)
- ⇨ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

Example:- 17

If $\vec{a} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{b} = -\hat{\imath} + 3\hat{k}$ then find $|\vec{a} \times \vec{b}|$

Ans: - We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$

$= \{(3\times3)-(0\times(-2))\}\hat{\imath} - \{(1\times3)-((-1)\times(-2))\}\hat{\jmath} + \{(1\times0)-((-1)\times3)\}\hat{k}$

$= 9\hat{\imath} - \hat{\jmath} + 3\hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{81+1+9} = \sqrt{91}$ (Ans)

Example:-18 Determine the area of the parallelogram whose adjacent sides are the vectors $\vec{a} = \widehat{2\imath}$ and $\vec{b} = 3\hat{\jmath}$. (2013-W)

Ans:- Area of the parallelogram with adjacent sides given by $\vec{a}$ and $\vec{b}$ is given by

area = $|\vec{a} \times \vec{b}| = |\widehat{2\imath} \times 3\hat{\jmath}| = |6\hat{k}| = 6$ sq units (Ans)

Example:-19 Find a unit vector perpendicular to both the vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = 3\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

Ans: - (2015-W and 2017-S)

Unit vector perpendicular to both $\vec{a}$and $\vec{b}$ is given by

$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$………………..(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 3 & -1 & 3 \end{vmatrix}$

$= (3-1)\hat{\imath} - (6+3)\hat{\jmath} + (-2-3)\hat{k}$

$= 2\hat{\imath} - 9\hat{\jmath} - 5\hat{k}$-----------------(2)

From (1) and (2) we have,

$\hat{n} = \frac{2\hat{\imath} - 9\hat{\jmath} - 5\hat{k}}{|2\hat{\imath} - 9\hat{\jmath} - 5\hat{k}|} = \frac{2\hat{\imath} - 9\hat{\jmath} - 5\hat{k}}{\sqrt{2^2+(-9)^2+(-5)^2}} = \frac{2\hat{\imath} - 9\hat{\jmath} - 5\hat{k}}{\sqrt{110}}$

$= \frac{2}{\sqrt{110}}\hat{\imath} - \frac{9}{\sqrt{110}}\hat{\jmath} - \frac{5}{\sqrt{110}}\hat{k}$ (ans)

Example:-20 If $\vec{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{b} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$, then find the sine of the angle between these vectors. (2016-w)

Ans :- We know that $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|.|\vec{b}|}$……………..(1)

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$= (1-4)\hat{\imath} - (-2-3)\hat{\jmath} + (8+3)\hat{k} = -3\hat{\imath} + 5\hat{\jmath} + 11\hat{k}$

Hence $\left|\vec{a} \times \vec{b}\right| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{9 + 25 + 121} = \sqrt{155}$ …………………(2)

Again $\left|\vec{a}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$ ………………………(3)

and $\left|\vec{b}\right| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$………………………..(4)

From equation (1),(2),(3) and (4) we have,

$\sin\theta = \frac{|\vec{a} \times \vec{b}\,|}{|\vec{a}|.|\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$ (Ans)

Q-21 Calculate the area of the triangle ABC (by vector method) where A(1,1,2), B(2,2,3) and C(3,-1,-1). (2013-W)

Solution: - Let the position vector of the vertices A,B and C is given by $\vec{a}\ ,\vec{b}\ and\ \vec{c}$ respectively.

Then $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\vec{b} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\vec{c} = 3\hat{\imath} - \hat{\jmath} - \hat{k}$

Now $\overrightarrow{AB} = Position\ vector\ of\ B - Position\ vector\ of\ A$

$= 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k} - (\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= (2-1)\hat{\imath} + (2-1)\hat{\jmath} + (3-2)\hat{k}$

$= \hat{\imath} + \hat{\jmath} + \hat{k}$

Similarly $\overrightarrow{AC} = Position\ vector\ of\ C - Position\ vector\ of\ A$

$= 3\hat{\imath} - \hat{\jmath} - \hat{k} - (\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$= (3-1)\hat{\imath} + (-1-1)\hat{\jmath} + (-1-2)\hat{k}$

$= 2\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -2 & -3 \end{vmatrix}$

$= (-3 + 2)\,\hat{\imath} - (-3 - 2)\,\hat{\jmath} + (-2 - 2)\,\hat{k} = -\hat{\imath} + 5\hat{\jmath} - 4\hat{k}$

Hence area of the triangle is given by

$\Delta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{(-1)^2 + 5^2 + (-4)^2}$

$= \frac{1}{2}\sqrt{1 + 25 + 16} = \frac{1}{2}\sqrt{42}$ sq units. (Ans)

Example:-22 Find the area of a parallelogram whose diagonals are determined by the vectors $\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$. (2014-W, 2017-W)

Ans: - Area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$ are given by

$$\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

Now $\vec{a} \times \vec{b}$ = $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

= (4 - 6) $\hat{\imath}$ - (12 + 2) $\hat{\jmath}$ + (-9 - 1) $\hat{k}$ = -2 $\hat{\imath}$ -14 $\hat{\jmath}$ - 10 $\hat{k}$

Now area $\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{(-2)^2 + (-14)^2 + (-10)^2}$

= $\frac{1}{2}\sqrt{4 + 196 + 100}$ = $\frac{\sqrt{300}}{2}$ = $\frac{10\sqrt{3}}{2}$ = 5 $\sqrt{3}$ sq unit. (ans)

Example:-23 For any vector $\vec{a}$ and $\vec{b}$, prove that $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a}\,.\vec{b})^2$ where a and b are magnitude of $\vec{a}$ and $\vec{b}$ respectively.

Proof: - $(\vec{a} \times \vec{b})^2$ = $(\mathrm{I}\,\vec{a}\,\mathrm{I}.\mathrm{I}\,\vec{b}\,\mathrm{I}\sin\theta\hat{n})^2$

= $(ab\sin\theta\hat{n})^2 = a^2b^2sin^2\theta$ (As $(\hat{n})^2$ =($|\hat{n}|$ $)^2$ = 1^2= 1)

= $a^2b^2(1 - cos^2\theta)$ = $a^2b^2 - a^2b^2cos^2\theta$

= $a^2b^2 - (abcos\theta)^2$ = $a^2b^2 - (\vec{a}\,.\vec{b})^2$ (Proved)

Example:-24 In a ΔABC , prove by vector method that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$,

where BC = a, CA = b and AB =c. (2017-S)

Proof:- As shown in figure- 25 ABC is a triangle having, $\vec{a} = \overrightarrow{BC}, \vec{b} = \overrightarrow{CA}$ and $\vec{c} = \overrightarrow{AB}$.

From triangle law of vector we know that ,

$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

⇨ $\vec{a} + \vec{b} = -\vec{c}$

⇨ $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ ………..(1)

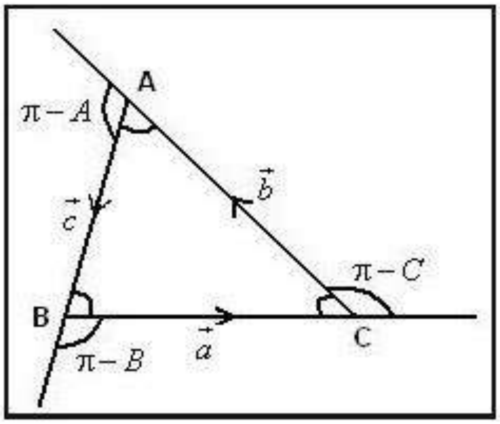


Fig-25

(taking cross product of both sides with $\vec{a}$ we get)

⇨ $\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

⇨ $(\vec{a} \times \vec{a}) + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $\vec{0} + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \vec{0}$

⇨ $(\vec{a} \times \vec{b}) = -(\vec{a} \times \vec{c})$

⇨ $\left(\vec{a}\times\vec{b}\right) = (\vec{c}\times\vec{a})$ --------------------(2)

Similarly taking cross product with $\vec{b}$ both sides of (1) we have,

⇨ $\left(\vec{a}\times\vec{b}\right) = (\vec{b}\times\vec{c})$ -----------------(3)

From (2) and (3) , $\left(\vec{a}\times\vec{b}\right) = (\vec{b}\times\vec{c}) = (\vec{c}\times\vec{a})$

⇨ $\left|\vec{a}\times\vec{b}\right| = \left|\vec{b}\times\vec{c}\right| = \left|\vec{c}\times\vec{a}\right|$

⇨ $ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

As from fig-25 it is clear that angle between $\vec{a}\ and\ \vec{b}$ is $\pi - C$, $\vec{b}\ and\ \vec{c}$ is $\pi - A$and $\vec{c}\ and\ \vec{a}$ is $\pi - B$.

Dividing above equation by abc we have,

⇨ $\frac{ab\sin(\pi-C)}{abc} = \frac{bc\sin(\pi-A)}{abc} = \frac{ca\sin(\pi-B)}{abc}$

⇨ $\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

Hence $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$ (Proved).

Example:-25 What inference can you draw when $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

Ans: - Given $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = \vec{0}$

⇨ { Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ II $\vec{b}$ } and { $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$ }

⇨ As $\vec{a}$ II $\vec{b}$ and $\vec{a} \perp \vec{b}$ cannot be hold simultaneously so $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

Hence either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

Example:-26 If $\left|\vec{a}\right| = 2$ and $\left|\vec{b}\right| = 5$ and $\left|\vec{a}\times\vec{b}\right| = 8$, then find $\vec{a}.\vec{b}$.

Ans: - Given $\left|\vec{a}\times\vec{b}\right| = 8$

⇨ $\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta = 8$

⇨ $2 \times 5 \sin\theta = 8$

⇨ $\sin\theta = \frac{8}{10} = \frac{4}{5}$

∴ $\cos\theta = \sqrt{1 - sin^2\theta} = \sqrt{1 - (\frac{4}{5})^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$Hence\ \vec{a}.\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta = 2 \times 5 \times \frac{3}{5} = 6$ (Ans)

Example:-27 Show that the vectors $\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$, $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$, and $4\hat{\imath} - 7\hat{\jmath} + 10\hat{k}$ are co-planar.(2017-S)

Ans: - Now let us find the following determinant ,

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = 1(-10+14) - (-3)(20-8) + 4(-14+4) = 4 + 36 - 40 = 0$$

Hence the three given vectors are co-planar.

Exercise

1. Show that the points (3,4) ,(1,7) and (-5,16) are collinear. (2 Marks)

2. If $\vec{a} = 3\hat{\imath} - 5\,\hat{\jmath}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath}$, then find the unit vector parallel to $\vec{a} + 2\vec{b}$. (2 Marks)

3. Show that the vectors $\vec{a} = 3\sqrt{3}\hat{\imath} - 3\,\hat{\jmath}$, $\vec{b} = 6\hat{\jmath}$ and $\vec{c} = 3\sqrt{3}\hat{\imath} + 3\,\hat{\jmath}$ form the sides of an equilateral triangle. (5 Marks)

4. Find the unit vector parallel to the sum $\vec{a} = 2\hat{\imath} + 4\,\hat{\jmath} - 5\hat{k}$ and $\vec{b} = \hat{\imath} + 2\,\hat{\jmath} + 3\hat{k}$. (2014-W,2017-W).(2 Marks)

5. Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$,where $\vec{a} = \hat{\imath} + \hat{\jmath}$ and $\vec{b} = \hat{\jmath} + \hat{k} - 2\hat{\imath}$.(2015-W)
(5 Marks)

6. The position vector of A,B and C are $2\hat{\imath} + \hat{\jmath} - \hat{k}, 3\hat{\imath} - 2\,\hat{\jmath} + \hat{k},\ \hat{\imath} + 4\,\hat{\jmath} - 3\hat{k}$ respectively . Show that A, B and C are collinear. (2 Marks)

7. Find the value of 'a' such that the vectors $\hat{\imath} - \hat{\jmath} + \hat{k},\ 2\hat{\imath} + \hat{\jmath} + \hat{k}\ and\ \ a\,\hat{\imath} - \hat{\jmath} + \mathrm{a}\hat{k}$are coplanar.
(2 Marks)

8. Find the value of 'k' so that the vectors $\vec{a} = \hat{\imath} + 2\,\hat{\jmath} - \hat{k}$ and $\vec{b} = \mathrm{k}\hat{\imath} + \hat{\jmath} + 5\hat{k}$ are perpendicular to each other. (2015-W) (2 Marks)

9.Find the unit vector in the direction of $2\vec{a} + 3\vec{b}$ where $\vec{a} = \hat{\imath} + 3\,\hat{\jmath} + \hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\,\hat{\jmath} - \hat{k}$.
(5 Marks)

10. Find the angle between the vectors $\vec{a} = 3\hat{\imath} + 2\,\hat{\jmath} - 6\hat{k}$ and $\vec{b} = 4\hat{\imath} - 3\,\hat{\jmath} + \hat{k}$. (5 Marks)

11. Calculate the area of the triangle ABC by vector method where A(1,2,4), B(3,1,-2) and C(4,3,1).
(5 Marks)

12. Obtain the area of the parallelogram whose adjacent sides are given by vectors $\hat{\imath} + 2\,\hat{\jmath} + 3\hat{k}$ and -3$\hat{\imath} - 2\,\hat{\jmath} + \hat{k}$. (5 Marks)

13. Determine the sine of the angle between $\vec{a} = \hat{\imath} - 3\,\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} + \hat{k}$. (5 Marks)

14. Find the unit vector along the direction of vector $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$. (2015-S) (2 Marks)

15. Find the area of the parallelogram having adjacent sides $\hat{\imath} - 2\,\hat{\jmath} + 2\hat{k}$ and $2\hat{\imath} + \hat{\jmath}$. (5 Marks)

16. Find the unit vector perpendicular to both $3\hat{\imath} + 2\,\hat{\jmath} - 3\hat{k}$ and $\hat{\imath} + \hat{\jmath} - \hat{k}$. (5 Marks)

17. Find the area of the parallelogram having vertices A(5,-1,1), B(-1,-3,4), C(1,-6,10) and D(7,-4,7).
(5 Marks)

18. Find the vector joining the points (2,-3) and (-1,1).Find its magnitude and the unit vector along the same direction. Also determine the component vectors along the co-ordinate axes. (5 Marks)

19. Prove by vector method , that in a triangle ABC,
$cosA = \frac{b^2+c^2-a^2}{2bc}$ where BC = a, CA = b and AB =c. (5 Marks)

20. Find the work done by the force $4\hat{\imath} - 3\hat{k}$ on a particle to displace it from (1,2,0) to (0,2,3) (2 Marks)

21. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then show that $(\vec{a} + \vec{b}\,)^2 = (\vec{a} - \vec{b}\,)^2$. (2 Marks)

22. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, prove that $(\vec{a}+\vec{b}+\overrightarrow{c})$ is equally inclined with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. (10 Marks)

23. Find the area of the parallelogram whose diagonals are $\vec{a} = 2\hat{\imath} - 3\,\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -3\hat{\imath} + 4\,\hat{\jmath} - \hat{k}$. (5 Marks)

Answers

2) $\frac{7}{5\sqrt{2}}\hat{\imath} + \frac{1}{5\sqrt{2}}\hat{\jmath}$

4) $\frac{3\hat{\imath}+6\,\hat{\jmath}-2\hat{k}}{7}$ 5) $\frac{-1}{\sqrt{6}}$, $\frac{-1}{6}(\hat{\jmath} + \hat{k} - 2\hat{\imath})$

7) 1

8) 3

9) $\frac{11}{\sqrt{122}}\hat{\imath} - \frac{1}{\sqrt{122}}\hat{k}$

10) $\frac{\pi}{2}$

11) $\frac{5\sqrt{10}}{2}$ sq units

12) $6\sqrt{5}$ sq units

13) $\frac{4\sqrt{2}}{\sqrt{33}}$

14) $\frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$

15) $\sqrt{45}$ sq units

16) $\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{k}$

17) $\sqrt{2257}$ sq units

18) $-3\hat{i} + 4\,\hat{j}$, 5 , $\frac{-3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$, $\frac{-3}{5}\hat{\imath}$ and $\frac{4}{5}\hat{\jmath}$.

20) -13 units

23) $\frac{3\sqrt{30}}{2}$ sq units.